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Derived decompositions of abelian categories are introduced in internal terms of abelian subcategories. They are used to construct semiorthogonal decompositions (or in other terminology, Bousfield localizations, or hereditary torsion pairs) in derived categories of abelian categories. A sufficient condition is given for abelian categories to have derived decompositions. This is necessary if abelian categories have enough projectives and injectives. Applications are given to homological ring epimorphisms, localizing subcategories, nonsingular rings and commutative noetherian rings. Moreover, a derived stratification of module categories over commutative noetherian rings of Krull dimension at most 1 is presented.

1. Introduction

Semiorthogonal decompositions (or hereditary torsion pairs in the terminology of [Beligiannis and Reiten 2007]) have been applied in a number of branches of mathematics. For example, in homotopy and triangulated categories, they were also named as Bousfield localizations [Neeman 2001, §9.1] and applied to get t-structures of triangulated categories (see [Beĭlinson et al. 1982]), and in algebraic geometry they were used to study Fourier–Mukai transforms on derived categories of coherent sheaves of smooth projective varieties (see [Huybrechts 2006, Chapter 11; Bondal and Orlov 1995]). In the course of studying semiorthogonal decompositions (see Definition 2.1), the following fundamental question seems to remain:

Question. Given an abelian category $\mathcal{A}$, how can we construct semiorthogonal decompositions of the $*$-bounded derived category $\mathscr{D}^*(\mathcal{A})$ of $\mathcal{A}$ for $* \in \{b, +, -, \varnothing\}$?

To answer this question, we characterize semiorthogonal decompositions of an abelian category directly in terms of abelian subcategories.

Definition 1.1. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{A}$. For $* \in \{b, +, -, \varnothing\}$, $\mathscr{D}^*(\mathcal{A})$ denotes the $*$-bounded derived category of $\mathcal{A}$. The pair $(\mathcal{X}, \mathcal{Y})$ is called a $\mathscr{D}^*$-*decomposition* of $\mathcal{A}$ if

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(D_1) both $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $\mathcal{A}$, and the inclusions $\mathcal{X} \subseteq \mathcal{A}$ and $\mathcal{Y} \subseteq \mathcal{A}$ induce fully faithful functors $\mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A})$ and $\mathscr{D}^*(\mathcal{Y}) \to \mathscr{D}^*(\mathcal{A})$, respectively.

(D_2) $\mathrm{Hom}_{\mathscr{D}^*(\mathcal{A})}(X, Y[n]) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $n \in \mathbb{Z}$.

(D_3) For each object $M^\bullet \in \mathscr{D}^*(\mathcal{A})$, there is a triangle $X_{M^\bullet} \to M^\bullet \to Y^{M^\bullet} \to X_{M^\bullet}[1]$ in $\mathscr{D}^*(\mathcal{A})$ such that $X_{M^\bullet} \in \mathscr{D}^*(\mathcal{X})$ and $Y^{M^\bullet} \in \mathscr{D}^*(\mathcal{Y})$.

Since the existence of $\mathscr{D}^b$-decompositions is the weakest condition among those other type of derived decompositions introduced in Definition 1.1, we sometimes pay more attention to the existence of such decompositions. For convenience, a $\mathscr{D}^b$-decomposition of $\mathcal{A}$ is termed *derived decomposition* of $\mathcal{A}$.

We present sufficient and necessary conditions of when $\mathscr{D}^*$-decompositions (and thus also semiorthogonal decompositions) of abelian categories exist. This is given in entirely internal terms of the abelian categories. We then apply our characterization to construct $\mathscr{D}^*$-decompositions for a wide variety of situations, including homological ring epimorphisms, localizing subcategories and commutative noetherian rings. Also, we show that the module category over a commutative noetherian ring of Krull dimension at most 1 has a derived stratification. But for an indecomposable commutative ring its derived category does not have nontrivial stratification by derived categories of rings (see [Angeleri Hügel et al. 2017]). Compared with this phenomenon, derived decompositions of abelian categories provide a new way to approach the derived category of an abelian category by those of its smaller abelian subcategories.

Our main result reads as follows.

Theorem 1.2. *Let $\mathcal{A}$ be an abelian category, $\mathcal{X}$ and $\mathcal{Y}$ full subcategories of $\mathcal{A}$ and $* \in \{b, +, -, \varnothing\}$.*

(1) *The pair $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$ if the following conditions hold*:
   - (a) $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$ *for any $n \geq 0$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.*
   - (b) *For each object $M \in \mathcal{A}$, there is a long exact sequence $0 \to Y_M \to X_M \to M \to Y^M \to X^M \to 0$ in $\mathcal{A}$ with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.*
   - (c) *For each object $M \in \mathcal{A}$, there is a monomorphism $M \to I$ in $\mathcal{A}$ such that $X^I = 0$ in* (*b*).
   - (d) *For each object $M \in \mathcal{A}$, there is an epimorphism $P \to M$ in $\mathcal{A}$ such that $Y_P = 0$ in* (*b*).

(2) *Suppose that $\mathcal{A}$ has enough projectives and injectives. Then $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$ if and only if* (*a*) *and* (*b*) *together with* (*c*′) *and* (*d*′) *hold, where*
   - (c′) *if M is injective, then $X^M = 0$ in* (*b*);
   - (d′) *if M is projective, then $Y_M = 0$ in* (*b*).

Note that (a) and (b) in Theorem 1.2 together are equivalent to saying that the pair $(\mathcal{X}, \mathcal{Y})$ is a *complete Ext-orthogonal pair* in $\mathcal{A}$, see (GC) and Lemma 2.3. Such pair was first introduced by Krause and Šťovíček [2010] to study the telescope conjecture for hereditary rings.

Theorem 1.2(2) implies that if $\mathcal{A}$ has enough projectives and injectives, then the existences of $\mathscr{D}^*$-decompositions of $\mathcal{A}$ for all $* \in \{b, +, -, \varnothing\}$ are equivalent. In particular, we have the following consequence.

Corollary 1.3. *Let* $\lambda : R \to S$ *be a homological ring epimorphism. Define* $\mathcal{Y} := \{Y \in R\text{-Mod} \mid \mathrm{Hom}_R(S, Y) = 0 = \mathrm{Ext}^1_R(S, Y)\}$ *and* $\mathcal{Z} := \{X \in R\text{-Mod} \mid S \otimes_R X = 0 = \mathrm{Tor}^R_1(S, X)\}$. *Then*

(1) $(S\text{-Mod}, \mathcal{Y})$ *is a derived decomposition of* R-Mod *if and only if* $\mathrm{projdim}({}_R S) \leq 1$ *and* $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), \mathrm{Ker}(\lambda)) = 0$.

(2) $(\mathcal{Z}, S\text{-Mod})$ *is a derived decomposition of* R-Mod *if and only if* $\mathrm{flatdim}(S_R) \leq 1$ *and* $\mathrm{Coker}(\lambda) \otimes_R I = 0$ *for any injective* R*-module* I.

(3) *If* (1) *and* (2) *are satisfied, then, for any* $* \in \{b, +, -, \varnothing\}$, *we have* $\mathscr{D}^*(\mathcal{Y}) \xrightarrow{\simeq} \mathscr{D}^*(\mathcal{Z})$ *and there exists a recollement*:

$$\mathscr{D}^*(S) \longrightarrow \mathscr{D}^*(R) \longrightarrow \mathscr{D}^*(\mathcal{Y}) \; .$$

In Corollary 1.3(3), the derived equivalences hold for arbitrary homological ring epimorphisms. They generalize the Matlis equivalences of derived categories of abelian categories in [Positselski 2018, Theorem 7.6] for localizations of commutative rings. Note that the derived equivalences of bounded derived categories in Corollary 1.3(3) have recently been extended to the ones of derived categories of other types in [Bazzoni and Positselski 2020]. Thus Corollary 1.3(3) coincides with [Bazzoni and Positselski 2020]. Moreover, our method developed in this paper is based on the technique of complete Ext-orthogonal pairs and can be applied to general abelian categories. For a general construction of half recollements of derived categories from derived decompositions, we refer the reader to Corollary 3.16.

Applying Theorem 1.2 to commutative rings, we have the following corollary. For notation and notions, we refer the reader to Section 4.4.

Corollary 1.4. *Let* R *be a commutative noetherian ring.*

(1) *Suppose that* Φ *is a specialization closed subset of* $\mathrm{Spec}(R)$ *and* $\Phi^c := \mathrm{Spec}(R) \setminus \Phi$. *Then*

  (i) $(\mathrm{Supp}^{-1}(\Phi), \mathrm{Supp}^{-1}(\Phi^c))$ *is a derived decomposition of* R-Mod *if and only if* Φ^c *is coherent*, *where* $\mathrm{Supp}^{-1}(\Phi)$ *is the category of those* R*-modules* M *such that their supports are contained in* Φ.

(ii) *If the Krull dimension of R is at most* 1, *then* $(\mathrm{Supp}^{-1}(\Phi), \mathrm{Supp}^{-1}(\Phi^c))$ *is a derived decomposition of* R-Mod*.*

(2) *Let* Σ *be a multiplicative subset of* R, S *the localization of* R *at* Σ *and* $\Phi := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap \Sigma \neq \varnothing\}$. *Then* $(\mathrm{Supp}^{-1}(\Phi), S\text{-Mod})$ *is a derived decomposition of* R-Mod*.*

Further application of Theorem 1.2 to localizing subcategories is given by Proposition 4.6. For a commutative noetherian ring of Krull dimension at most 1, we show that its module category always admits a derived stratification (see Corollary 4.11). For a left nonsingular ring, we have the following result. Examples of left nonsingular rings include left semihereditary rings, direct products of integral domains, semiprime left Goldie rings and commutative semiprime rings (see [Goodearl 1976]).

Corollary 1.5. *Let* R *be a ring*, $\mathcal{X}$ *be the full subcategory of singular modules in* R-Mod, *and* $\mathcal{Y}$ *be the full subcategory of* R-Mod *consisting of all direct summands of arbitrary products of copies of the injective envelope of* ${}_RR$. *If* R *is left nonsingular*, *then* $(\mathcal{X}, \mathcal{Y})$ *is a derived decomposition of* R-Mod*.*

The article is outlined as follows: In Section 2 we fix notation and recall definitions needed in proofs. In Section 3 we prove Theorem 1.2. The proof is divided into two parts. The first one is for the proof of Theorem 1.2(1), while the second one is for that of Theorem 1.2(2). In Section 4 we apply Theorem 1.2 to construct derived decompositions of module categories from various aspects: ring epimorphisms, localizing subcategories, nonsingular ring and commutative noetherian rings, and therefore prove the three corollaries.

In the second paper we shall construct complete cotorsion pairs from derived decompositions and then apply them to infinitely generated tilting modules, and also establish inequalities of homological dimensions of abelian categories involved in derived decompositions.

2. Notation and definitions

In this section we first fix some notation and recall definitions of semiorthogonal decompositions (or hereditary torsion pairs), cotorsion pairs and complete Ext-orthogonal pairs.

2.1. ***Notation for derived categories.*** Let $\mathcal{A}$ be an additive category.

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is always assumed to be closed under isomorphisms. For an object $X \in \mathcal{A}$, $\mathrm{add}(X)$ (respectively, $\mathrm{Add}(X)$) denotes the full subcategory of $\mathcal{A}$ consisting of all direct summands of finite (respectively, arbitrary) coproducts of copies of X (if arbitrary coproducts exist).

Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor from $\mathcal{A}$ to another additive category $\mathcal{A}'$. The kernel and image of F are defined as $\mathrm{Ker}(F) := \{X \in \mathcal{A} \mid FX \simeq 0\}$ and $\mathrm{Im}(F) := \{Y \in \mathcal{A}' \mid \text{there exists } X \in \mathcal{A}, FX \simeq Y\}$, respectively. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. The kernel, image and cokernel of f, whenever they exist, will be denoted by $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ and $\mathrm{Coker}(f)$, respectively.

By a complex $X^\bullet$ over $\mathcal{A}$ we mean a sequence of morphisms d^i between objects X^i in $\mathcal{A}$:

$$\cdots \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \to \cdots,$$

such that $d^i d^{i+1} = 0$ for all $i \in \mathbb{Z}$. We write $X^\bullet = (X^i, d^i)_{i\in\mathbb{Z}}$ and call d^i the i-th differential of $X^\bullet$. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting n degrees, that is, $(X^\bullet[n])^i = X^{n+i}$ with the i-th differential $(-1)^n d^{n+i}$, and by $H^n(X^\bullet)$ the n-th cohomology of $X^\bullet$.

Let $\mathscr{C}(\mathcal{A})$ be the category of all complexes over $\mathcal{A}$ with chain maps as morphisms, and $\mathscr{K}(\mathcal{A})$ the homotopy category of $\mathscr{C}(\mathcal{A})$. We denote by $\mathscr{C}^b(\mathcal{A})$ and $\mathscr{K}^b(\mathcal{A})$ the bounded complex and homotopy categories of $\mathcal{A}$, respectively.

From now on, let $\mathcal{A}$ be an abelian category.

By $\mathscr{D}(\mathcal{A})$ and $\mathscr{D}^b(\mathcal{A})$ we denote the *unbounded* and *bounded derived categories* of $\mathcal{A}$, respectively. Throughout the paper, we always identify $\mathscr{D}^b(\mathcal{A})$ with the full subcategory of $\mathscr{D}(\mathcal{A})$ consisting of all complexes with finitely many nonzero cohomologies because they are equivalent as triangulated categories. Further, by $\mathscr{D}^+(\mathcal{A})$ and $\mathscr{D}^-(\mathcal{A})$ we denote the *bounded-below* and *bounded-above derived categories* of $\mathcal{A}$, respectively.

For $X, Y \in \mathcal{A}$ and $i \in \mathbb{Z}$, we write $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ for $\mathrm{Hom}_{\mathscr{D}(\mathcal{A})}(X, Y[i])$. Note that $\mathrm{Ext}^0_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$ whenever $i < 0$. For each $i \geq 1$, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ can be identified with the set of equivalence classes of long exact sequences $0 \to Y \to E_i \to \cdots \to E_1 \to X \to 0$ in $\mathcal{A}$ (see [Iversen 1986, XI] for details).

The following facts are standard in homological algebra.

(1) Suppose that $\mathcal{A}$ has enough projectives with $\mathscr{P}(\mathcal{A})$ the category of all projective objects of $\mathcal{A}$. Further, let $\mathscr{K}^{-,b}(\mathscr{P}(\mathcal{A}))$ be the full subcategory of $\mathscr{K}(\mathcal{A})$ consisting of bounded-above complexes with all terms in $\mathscr{P}(\mathcal{A})$ and finitely many nonzero cohomologies. Then there is a triangle equivalence between $\mathscr{K}^{-,b}(\mathscr{P}(\mathcal{A}))$ and $\mathscr{D}^b(\mathcal{A})$. In this case, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ is isomorphic to the usual i-th extension group of X and Y, defined by projective resolutions of X.

(2) Dually, suppose that $\mathcal{A}$ has enough injectives with $\mathscr{I}(\mathcal{A})$ the category of all injective objects of $\mathcal{A}$. Then there is a triangle equivalence between $\mathscr{K}^{+,b}(\mathscr{I}(\mathcal{A}))$ and $\mathscr{D}^b(\mathcal{A})$, where $\mathscr{K}^{+,b}(\mathscr{I}(\mathcal{A}))$ is defined similarly. In this situation, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ can be calculated by taking injective resolutions of Y.

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an *abelian subcategory* of $\mathcal{A}$ if $\mathcal{B}$ is an abelian category and the inclusion $\mathcal{B} \to \mathcal{A}$ is an exact functor between abelian categories. This is equivalent to saying that $\mathcal{B}$ is closed under taking kernels and cokernels in $\mathcal{A}$. The full subcategories $\{0\}$ and $\mathcal{A}$ are called the *trivial* abelian subcategories of $\mathcal{A}$.

Let $\mathbb{N}$ denote the set of nonnegative integers. For $n \in \mathbb{N}$ and a full subcategory $\mathcal{B}$ of $\mathcal{A}$, we define the full subcategories of $\mathcal{A}$:

$$\begin{aligned} {}^{\perp n}\mathcal{B} &:= \{X \in \mathcal{A} \mid \mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0, Y \in \mathcal{B}\}, \\ {}^{\perp > n}\mathcal{B} &:= \{X \in \mathcal{A} \mid \mathrm{Ext}^j_{\mathcal{A}}(X, Y) = 0, Y \in \mathcal{B}, j > n\}, \\ {}^{\perp}\mathcal{B} &:= \bigcap_{n \in \mathbb{N}} {}^{\perp n}\mathcal{B}. \end{aligned}$$

Similarly, $\mathcal{B}^{\perp n}$, $\mathcal{B}^{\perp > n}$ and $\mathcal{B}^{\perp}$ are defined. Recall that ${}^{\perp}\mathcal{B}$ is said to be *left perpendicular* to $\mathcal{B}$ in $\mathcal{A}$, while $\mathcal{B}^{\perp}$ is said to be *right perpendicular* to $\mathcal{B}$ in $\mathcal{A}$ (see [Geigle and Lenzing 1991]).

Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor of abelian categories. Then F induces derived functors $\mathcal{D}^*(F) : \mathcal{D}^*(\mathcal{A}) \to \mathcal{D}^*(\mathcal{A}')$ for any $* \in \{b, +, -, \varnothing\}$, defined by $F(X^\bullet) := (FX^i, Fd^i)_{i \in \mathbb{Z}}$ for $X^\bullet \in \mathcal{D}^*(\mathcal{A})$.

By a ring we mean an associative ring R with identity. We denote by R-Mod the category of all unitary left R-modules. For an R-module M, we denote by $\mathrm{projdim}({}_RM)$, $\mathrm{injdim}({}_RM)$ and $\mathrm{flatdim}({}_RM)$ the projective, injective and flat dimensions of M, respectively. As usual, we simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for the complex, homotopy and derived categories of R-Mod, respectively.

Let $\lambda : R \to S$ be a homomorphism of rings. We denote by $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ the *restriction functor* induced by λ, and by $\mathscr{D}(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ the derived functor of λ_*. If λ_* is fully faithful, then λ is called a *ring epimorphism*. If $\mathscr{D}(\lambda_*)$ is fully faithful, then λ is called a *homological* ring epimorphism. Note that λ is a homological ring epimorphism if and only if the multiplication $S \otimes_R S \to S$ is an isomorphism and $\mathrm{Tor}^R_n(S, S) = 0$ for all $n \geq 1$. In this case, we identify S-Mod with $\mathrm{Im}(\lambda_*)$, and $\mathscr{D}(S)$ with $\mathrm{Im}(\mathscr{D}(\lambda_*))$.

2.2. ***Semiorthogonal decompositions and half recollements.*** Semiorthogonal decompositions are also called hereditary torsion pairs in triangulated categories (see [Neeman 2001, §9.1; Beligiannis and Reiten 2007, Chapter I.2]). The precise definition reads as follows:

Definition 2.1. Let $\mathscr{D}$ be a triangulated category with a shift functor [1]. A pair $(\mathscr{X}, \mathscr{Y})$ of full subcategories $\mathscr{X}$ and $\mathscr{Y}$ of $\mathscr{D}$ is called a *semiorthogonal decomposition* of $\mathscr{D}$ if

(1) $\mathscr{X}$ and $\mathscr{Y}$ are triangulated subcategories of $\mathscr{D}$.

(2) $\mathrm{Hom}_{\mathscr{D}}(X, Y) = 0$ for all $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

(3) For each object $D \in \mathscr{D}$, there is a triangle $X_D \to D \to Y^D \to X_D[1]$ in $\mathscr{D}$ with $X_D \in \mathscr{X}$ and $Y^D \in \mathscr{Y}$.

Semiorthogonal decompositions are closely related to half recollements of triangulated categories. In fact, a pair $(\mathscr{X}, \mathscr{Y})$ of full triangulated subcategories of $\mathscr{D}$ is a semiorthogonal decomposition of $\mathscr{D}$ if and only if there exists a *lower half recollement* among $\mathscr{X}$, $\mathscr{D}$ and $\mathscr{Y}$, in the sense that there are four triangle functors demonstrated in the diagram

$$\mathscr{X} \underset{\boldsymbol{R}}{\overset{\boldsymbol{i}}{\rightleftarrows}} \mathscr{D} \underset{\boldsymbol{j}}{\overset{\boldsymbol{L}}{\rightleftarrows}} \mathscr{Y}$$

such that

(1) $\boldsymbol{i}$ and $\boldsymbol{j}$ are canonical inclusions;

(2) both $(\boldsymbol{i}, \boldsymbol{R})$ and $(\boldsymbol{L}, \boldsymbol{j})$ are adjoint pairs;

(3) $\boldsymbol{L}\boldsymbol{i} = 0$;

(4) for each object $D \in \mathscr{D}$, there is a distinguished triangle $\boldsymbol{i}\boldsymbol{R}(D) \to D \to \boldsymbol{j}\boldsymbol{L}(D) \to \boldsymbol{i}\boldsymbol{R}(D)[1]$ in $\mathscr{D}$, where $\boldsymbol{i}\boldsymbol{R}(D) \to D$ is the counit adjunction and $D \to \boldsymbol{j}\boldsymbol{L}(D)$ is the unit adjunction.

In this case, there are equivalences of triangulated categories: $\mathscr{D}/\mathscr{X} \xrightarrow{\simeq} \mathscr{Y}$ and $\mathscr{D}/\mathscr{Y} \xrightarrow{\simeq} \mathscr{X}$. Observe that the conditions in Definition 2.1 are weaker than the ones given in [Böhning et al. 2014; Huybrechts 2006; Orlov 2009] because $\boldsymbol{i}$ may not have a left adjoint, nor $\boldsymbol{j}$ have a right adjoint. But, if $\boldsymbol{i}$ does have a left adjoint (or equivalently, $\boldsymbol{L}$ has a fully faithful left adjoint), then the lower half recollement can be completed to a *recollement* among triangulated categories $\mathscr{X}$, $\mathscr{D}$ and $\mathscr{Y}$ in the sense of Beilinson, Bernstein and Deligne (see [Beĭlinson et al. 1982] for definition). Recollements of derived module categories appear often in studying infinitely generated tilting modules (for example, see [Chen and Xi 2012; 2019]).

2.3. ***Ext-orthogonal pairs in abelian categories.*** Derived decompositions of abelian categories are associated with both complete cotorsion pairs and complete Ext-orthogonal pairs in abelian categories. The notion of complete cotorsion pairs is classical and has been widely applied to relative homological algebra and generalized tilting theory (see [Enochs and Jenda 2000; Beligiannis and Reiten 2007; Hovey 2002]), while the notion of complete Ext-orthogonal pairs seems only to be employed in dealing with the telescope conjecture for hereditary rings (see [Krause and Šťovíček 2010]). We will show in the next section that the latter may be useful in derived decompositions.

Throughout this section, $\mathcal{A}$ is an abelian category, and $(\mathcal{X}, \mathcal{Y})$ is a pair of full subcategories of $\mathcal{A}$.

Definition 2.2 [Krause and Šťovíček 2010, Definition 2.1]. The pair $(\mathcal{X}, \mathcal{Y})$ is said to be *Ext-orthogonal* in $\mathcal{A}$ if $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$; and *complete Ext-orthogonal* in $\mathcal{A}$ if it is Ext-orthogonal and satisfies the gluing condition

(GC) For each object $M \in \mathcal{A}$, there exists a 5-term exact sequence in $\mathcal{A}$

$$\varepsilon_M : \ 0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

The following result implies that the two conditions (a) and (b) in Theorem 1.2 hold if and only if $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$.

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{A}$. If $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$ and $(\mathcal{X}, \mathcal{Y})$ satisfies (GC), then*

(1) $\mathcal{X} = {}^{\perp_0}\mathcal{Y} \cap {}^{\perp_1}\mathcal{Y}$ *and* $\mathcal{Y} = \mathcal{X}^{\perp_0} \cap \mathcal{X}^{\perp_1}$.

(2) $(\mathcal{X}, \mathcal{Y})$ *is complete Ext-orthogonal.*

(3) *Both $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $\mathcal{A}$.*

Proof. (1) It follows from $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$ that $\mathcal{X} \subseteq {}^{\perp_0}\mathcal{Y} \cap {}^{\perp_1}\mathcal{Y}$. Now, we show the converse of this inclusion. Let $M \in {}^{\perp_0}\mathcal{Y} \cap {}^{\perp_1}\mathcal{Y}$. Since $(\mathcal{X}, \mathcal{Y})$ satisfies (GC), there is a five-term exact sequence ε_M for M in $\mathcal{A}$. In particular, both Y_M and Y^M belong to $\mathcal{Y}$. It then follows from $\mathrm{Hom}_{\mathcal{A}}(M, Y^M) = 0$ that there is a short exact sequence $0 \to Y_M \to X_M \to M \to 0$ in $\mathcal{A}$. Further, this sequence splits due to $\mathrm{Ext}^1_{\mathcal{A}}(M, Y_M) = 0$. Thus $X_M \simeq Y_M \oplus M$. Since $\mathrm{Hom}_{\mathcal{A}}(X_M, Y_M) = 0$, we have $Y_M = 0$ and $M \simeq X_M \in \mathcal{X}$. So ${}^{\perp_0}\mathcal{Y} \cap {}^{\perp_1}\mathcal{Y} \subseteq \mathcal{X}$. Thus the first equality in (1) holds. Similarly, the second equality in (1) holds.

(2) It suffices to show both $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. But this follows from (1) and the inclusion $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$.

(3) We first prove that $\mathcal{X}$ is an abelian subcategory of $\mathcal{A}$. Clearly, $\mathcal{X}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $\mathcal{A}$. Thus $\mathcal{X}$ is an abelian subcategory of $\mathcal{A}$ if and only if $\mathcal{X}$ is closed under cokernels (or equivalently, kernels) in $\mathcal{A}$. But the latter follows from the dual statement of [Geigle and Lenzing 1991, Proposition 1.1].

The conclusion on $\mathcal{Y}$ is an immediate consequence of [Geigle and Lenzing 1991, Proposition 1.1]. □

Lemma 2.4 [Krause and Šťovíček 2010, Lemma 2.9]. *Let $(\mathcal{X}, \mathcal{Y})$ be an Ext-orthogonal pair in an abelian category $\mathcal{A}$ and M an object in $\mathcal{A}$. Suppose that there is an exact sequence*

$$\varepsilon_M : 0 \longrightarrow Y_M \xrightarrow{\varepsilon_M^{-2}} X_M \xrightarrow{\varepsilon_M^{-1}} M \xrightarrow{\varepsilon_M^{0}} Y^M \xrightarrow{\varepsilon_M^{1}} X^M \longrightarrow 0$$

in $\mathcal{A}$ with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

(1) *There are isomorphisms of abelian groups for all* $X \in \mathcal{X}$ *and* $Y \in \mathcal{Y}$:

$$(\varepsilon_M^{-1})^* : \mathrm{Hom}_{\mathcal{A}}(X, X_M) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}}(X, M)$$

and

$$(\varepsilon_M^{0})_* : \mathrm{Hom}_{\mathcal{A}}(Y^M, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}}(M, Y).$$

(2) *If* $\varepsilon_N : 0 \to Y_N \xrightarrow{\varepsilon_N^{-2}} X_N \xrightarrow{\varepsilon_N^{-1}} N \xrightarrow{\varepsilon_N^{0}} Y^N \xrightarrow{\varepsilon_N^{1}} X^N \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $X_N, X^N \in \mathcal{X}$ *and* $Y_N, Y^N \in \mathcal{Y}$, *then each morphism* $f : M \to N$ *extends uniquely to a morphism* $\varepsilon_f : \varepsilon_M \to \varepsilon_N$ *of exact sequences*:

$$\begin{array}{ccccccccccccc}
\varepsilon_M : & & 0 & \longrightarrow & Y_M & \xrightarrow{\varepsilon_M^{-2}} & X_M & \xrightarrow{\varepsilon_M^{-1}} & M & \xrightarrow{\varepsilon_M^{0}} & Y^M & \xrightarrow{\varepsilon_M^{1}} & X^M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varepsilon_f} & & & & \downarrow{\scriptstyle Y_f} & & \downarrow{\scriptstyle X_f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Y^f} & & \downarrow{\scriptstyle X^f} & & \\
\varepsilon_N : & & 0 & \longrightarrow & Y_N & \xrightarrow{\varepsilon_N^{-2}} & X_N & \xrightarrow{\varepsilon_N^{-1}} & N & \xrightarrow{\varepsilon_N^{0}} & Y^N & \xrightarrow{\varepsilon_N^{1}} & X^N & \longrightarrow & 0.
\end{array}$$

(3) *Any exact sequence* $0 \to Y \to X \to M \to Y' \to X' \to 0$ *in* $\mathcal{A}$ *with* $X, X' \in \mathcal{X}$ *and* $Y, Y' \in \mathcal{Y}$ *is isomorphic to* ε_M.

Now, let $(\mathcal{X}, \mathcal{Y})$ be a complete Ext-orthogonal pair in an abelian category $\mathcal{A}$. For each object $M \in \mathcal{A}$, we fix an exact sequence in $\mathcal{A}$:

$$(*) \qquad \varepsilon_M : \quad 0 \longrightarrow Y_M \xrightarrow{\varepsilon_M^{-2}} X_M \xrightarrow{\varepsilon_M^{-1}} M \xrightarrow{\varepsilon_M^{0}} Y^M \xrightarrow{\varepsilon_M^{1}} X^M \longrightarrow 0$$

such that $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$. In particular, if $M \in \mathcal{X}$, then $\varepsilon_M^{-1} : X_M \to M$ is an isomorphism and $Y^M \simeq 0$; if $M \in \mathcal{Y}$, then $\varepsilon_M^{0} : M \to Y^M$ is an isomorphism and $X_M \simeq 0$.

By Lemma 2.3, both $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $\mathcal{A}$ closed under direct summands. Let $i : \mathcal{X} \to \mathcal{A}$ and $j : \mathcal{Y} \to \mathcal{A}$ be the inclusions. Then i and j are exact functors. Moreover, i has a right adjoint $r : \mathcal{A} \to \mathcal{X}$ and j has a left adjoint $\ell : \mathcal{A} \to \mathcal{Y}$, which are defined as follows:

For each $M \in \mathcal{A}$ and for a morphism $f : M \to N$ in $\mathcal{A}$,

$$r(M) = X_M, \quad r(f) = X_f : X_M \to X_N$$

and

$$\ell(M) = Y^M, \quad \ell(f) = Y^f : Y^M \to Y^N.$$

These are well defined by Lemma 2.4. For the adjoint pair (i, r) of functors, the unit adjunction of $X \in \mathcal{X}$ is given by the inverse of the isomorphism $\varepsilon_X^{-1} : r(X) \to X$, and the counit adjunction of $M \in \mathcal{A}$ is given by $\varepsilon_M^{-1} : ir(M) \to M$. Similarly, the unit and counit adjunctions associated with (ℓ, j) can be defined by ε_M^{0}.

Now, we can form the diagram of functors between abelian categories:

$$(\sharp)\qquad \mathcal{X} \underset{r}{\overset{i}{\rightleftarrows}} \mathcal{A} \underset{j}{\overset{\ell}{\rightleftarrows}} \mathcal{Y}$$

where r is left exact and ℓ is right exact. In general, neither r nor ℓ is exact. So $(\sharp)$ is neither a localization nor a colocalization sequence of abelian categories, and therefore, it may not be completed into a recollement of abelian categories. However, since i and j are exact, they induce derived functors between bounded derived categories: $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$ and $\mathscr{D}^b(j) : \mathscr{D}^b(\mathcal{Y}) \to \mathscr{D}^b(\mathcal{A})$. With the notation in $(\sharp)$, the sequence $(*)$ can be rewritten as follows:

$$(*)\qquad \varepsilon_M : 0 \longrightarrow Y_M \xrightarrow{\varepsilon_M^{-2}} r(M) \xrightarrow{\varepsilon_M^{-1}} M \xrightarrow{\varepsilon_M^{0}} \ell(M) \xrightarrow{\varepsilon_M^{1}} X^M \longrightarrow 0.$$

Finally, we consider the following full subcategories of $\mathcal{A}$ defined via the sequence $(*)$:

$$\mathcal{A}_{r\text{-}adj} := \{M \in \mathcal{A} \mid X^M = 0\} \quad \text{and} \quad \mathcal{A}_{\ell\text{-}adj} := \{M \in \mathcal{A} \mid Y_M = 0\}.$$

The following properties of the two subcategories will be used in the proof of Theorem 1.2(1).

Lemma 2.5. (1) *$\mathcal{A}_{r\text{-}adj}$ is closed under extensions and quotients in $\mathcal{A}$.*

(2) *$\mathcal{A}_{\ell\text{-}adj}$ is closed under extensions and subobjects in $\mathcal{A}$.*

(3) *The restriction of r to $\mathcal{A}_{r\text{-}adj}$ is exact, that is, if $0 \to M^{-2} \to M^{-1} \to M^0 \to 0$ is an exact sequence in $\mathcal{A}$ with $M^i \in \mathcal{A}_{r\text{-}adj}$ for $-2 \le i \le 0$, then $0 \to r(M^{-2}) \to r(M^{-1}) \to r(M^0) \to 0$ is an exact sequence in $\mathcal{X}$.*

(4) *The restriction of ℓ to $\mathcal{A}_{\ell\text{-}adj}$ is exact, that is, if $0 \to M^{-2} \to M^{-1} \to M^0 \to 0$ is an exact sequence in $\mathcal{A}$ with $M^i \in \mathcal{A}_{\ell\text{-}adj}$ for $-2 \le i \le 0$, then $0 \to \ell(M^{-2}) \to \ell(M^{-1}) \to \ell(M^0) \to 0$ is an exact sequence in $\mathcal{Y}$.*

Proof. We only prove (1) and (3) since (2) and (4) can be proved dually.

Let

$$0 \to M^{-2} \xrightarrow{f} M^{-1} \xrightarrow{g} M^0 \to 0$$

be an exact sequence in $\mathcal{A}$. We regard it as a complex $M^\bullet$ in $\mathscr{C}^b(\mathcal{A})$ with M^0 in degree 0. It follows from Lemma 2.4(2) that the sequence $(*)$, associated with M^i for $-2 \le i \le 0$, induces an exact sequence of complexes over $\mathcal{A}$:

$$0 \longrightarrow Y_{M^\bullet} \xrightarrow{\varepsilon_{M^\bullet}^{-2}} r(M^\bullet) \xrightarrow{\varepsilon_{M^\bullet}^{-1}} M^\bullet \xrightarrow{\varepsilon_{M^\bullet}^{0}} \ell(M^\bullet) \xrightarrow{\varepsilon_{M^\bullet}^{1}} X^{M^\bullet} \to 0,$$

where $r(M^\bullet)$ means $(r(M^i))_{i\in\mathbb{Z}}$. Recall that r is a left exact functor and ℓ is a right exact functor. Thus the complexes $r(M^\bullet)$ and $\ell(M^\bullet)$ are exact everywhere except in the degrees 0 and -2, respectively. Since $M^\bullet$ is an exact sequence, (1) holds.

For (3), suppose $M^i \in \mathcal{A}_{r\text{-}adj}$ for $-2 \le i \le 0$. Then $X^{M^\bullet} = 0$. To show that $r(M^\bullet)$ is a exact sequence, it suffices to show that the homomorphism $r(g) : r(M^{-1}) \to r(M^0)$ is surjective (or equivalently, $H^0(r(M^\bullet)) = 0$).

Let $c(M^\bullet)$ be the cokernel of the chain map $\varepsilon^{-2}_{M^\bullet}$. Taking cohomologies on the sequence $0 \to Y_{M^\bullet} \to r(M^\bullet) \to c(M^\bullet) \to 0$, we get a long exact sequence in $\mathcal{A}$:

$$H^{-1}(r(M^\bullet)) \longrightarrow H^{-1}(c(M^\bullet)) \longrightarrow H^0(Y_{M^\bullet}) \longrightarrow H^0(r(M^\bullet)) \longrightarrow H^0(c(M^\bullet)).$$

Note that $H^{-1}(r(M^\bullet)) = 0$ and $H^0(Y_{M^\bullet}) \simeq \mathrm{Coker}(Y_g)$, where $Y_g : Y_{M^{-1}} \to Y_{M^0}$ is induced from g. Since the sequence $M^\bullet$ is exact and $0 \to c(M^\bullet) \to M^\bullet \to \ell(M^\bullet) \to 0$ is exact in $\mathscr{C}^b(\mathcal{A})$, we have $H^i(c(M^\bullet)) \simeq H^{i-1}(\ell(M^\bullet))$ for any $i \in \mathbb{Z}$. This implies $H^0(c(M^\bullet)) \simeq H^{-1}(\ell(M^\bullet)) = 0$ and $H^{-1}(c(M^\bullet)) \simeq H^{-2}(\ell(M^\bullet)) \simeq \mathrm{Ker}(Y^f)$, where $Y^f : \ell(M^{-2}) \to \ell(M^{-1})$ is induced from f. Consequently, there is a short exact sequence in $\mathcal{A}$:

$$0 \longrightarrow \mathrm{Ker}(Y^f) \longrightarrow \mathrm{Coker}(Y_g) \longrightarrow H^0(r(M^\bullet)) \longrightarrow 0.$$

Clearly, $\mathrm{Ker}(Y^f), \mathrm{Coker}(Y_g) \in \mathcal{Y}$ and $H^0(r(M^\bullet)) \in \mathcal{X}$ since $\mathcal{Y}$ and $\mathcal{X}$ are abelian full subcategories of $\mathcal{A}$ by Lemma 2.3(3). Thus $H^0(r(M^\bullet)) \in \mathcal{X} \cap \mathcal{Y}$. It follows from $\mathcal{X} = {}^{\perp}\mathcal{Y}$ that $H^0(r(M^\bullet)) = 0$. So (3) holds. □

3. Derived decompositions of abelian categories

In this section we will prove Theorem 1.2. In particular, we show that a complete Ext-orthogonal pair is a derived decomposition of an abelian category with enough projectives and injectives if and only if the five-term exact sequences for both projective and injective objects are reduced to four terms.

3.1. ***Sufficient conditions for the existence of derived decompositions.*** Complete Ext-orthogonal pairs and derived decompositions are defined in terms of abelian categories. We will show that derived decompositions induce complete Ext-orthogonal pairs.

Proposition 3.1. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{A}$. Given $* \in \{b, +, -, \varnothing\}$, if $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$, then it is a complete Ext-orthogonal pair in $\mathcal{A}$.*

Proof. Suppose $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-derived decomposition of $\mathcal{A}$. By Definition 1.1(D2), $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$, and by Definition 1.1(D3), for each $M \in \mathcal{A}$, there is a triangle $X^\bullet \to M \to Y^\bullet \to X^\bullet[1]$ in $\mathscr{D}^*(\mathcal{A})$ such that $X^\bullet \in \mathscr{D}^*(\mathcal{X})$ and $Y^\bullet \in \mathscr{D}^*(\mathcal{Y})$, where $\mathscr{D}^*(\mathcal{X})$ and $\mathscr{D}^*(\mathcal{Y})$ can be regarded as triangulated subcategories of $\mathscr{D}^*(\mathcal{A})$ by $(D1)$ in Definition 1.1. By taking cohomologies on this triangle, one gets the long exact sequence $0 \to H^{-1}(Y^\bullet) \to H^0(X^\bullet) \to M \to H^0(Y^\bullet) \to H^1(X^\bullet) \to 0$ in $\mathcal{A}$. Recall that $\mathcal{X}$ is an abelian subcategory of $\mathcal{A}$. Since $X^\bullet \in \mathscr{D}^*(\mathcal{X})$, we have $H^i(X^\bullet) \in \mathcal{X}$ for all $i \in \mathbb{Z}$. Particularly, both $H^0(X^\bullet)$ and $H^1(X^\bullet)$ lie in $\mathcal{X}$. Similarly, both $H^{-1}(Y^\bullet)$ and $H^0(Y^\bullet)$ lie in $\mathcal{Y}$. Thus the pair $(\mathcal{X}, \mathcal{Y})$ satisfies the gluing condition. Now, by Lemma 2.3(2), $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$. □

Having shown that $\mathscr{D}^*$-decompositions are complete Ext-orthogonal pairs, we now consider its converse:

> Given a complete Ext-orthogonal pair $(\mathcal{X}, \mathcal{Y})$ in an abelian category $\mathcal{A}$, when is it a $\mathscr{D}^*$-decomposition of $\mathcal{A}$ for $* \in \{b, +, -, \varnothing\}$?

To address this question, we use derived categories of exact categories. Recall that an *exact category* (in the sense of Quillen) is an additive category $\mathcal{E}$ endowed with a class of conflations closed under isomorphism and satisfying certain axioms (for example, see [Keller 1996, §4]). In case that $\mathcal{E}$ is an abelian category, the class of conflations coincides with the class of short exact sequences.

Let $\mathcal{E}$ be a full subcategory of the abelian category $\mathcal{A}$. Suppose that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, that is, for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with both $X, Z \in \mathcal{E}$, we have $Y \in \mathcal{E}$. Then $\mathcal{E}$ endowed with the short exact sequences of $\mathcal{A}$ having their terms in $\mathcal{E}$ is an exact category and the inclusion $\mathcal{E} \subseteq \mathcal{A}$ is a fully faithful exact functor. Thus $\mathcal{E}$ is called a *fully exact subcategory* of $\mathcal{A}$.

A complex $X^\bullet \in \mathscr{C}(\mathcal{E})$ is said to be *strictly exact* if it is exact in $\mathscr{C}(\mathcal{A})$ and all of its boundaries belong to $\mathcal{E}$. Let $\mathscr{K}_{\mathrm{ac}}(\mathcal{E})$ be the full subcategory of $\mathscr{K}(\mathcal{E})$ consisting of those complexes which are isomorphic to strictly exact complexes. Then $\mathscr{K}_{\mathrm{ac}}(\mathcal{E})$ is a full triangulated subcategory of $\mathscr{K}(\mathcal{E})$. The *unbounded derived category* of $\mathcal{E}$, denoted by $\mathscr{D}(\mathcal{E})$, is defined to be the Verdier quotient of $\mathscr{K}(\mathcal{E})$ by $\mathscr{K}_{\mathrm{ac}}(\mathcal{E})$. Similarly, the bounded-below, bounded-above and bounded derived categories $\mathscr{D}^+(\mathcal{E})$, $\mathscr{D}^-(\mathcal{E})$ and $\mathscr{D}^b(\mathcal{E})$ can be defined through bounded-below, bounded-above and bounded complexes over $\mathcal{E}$, respectively. Moreover, the canonical functor $\mathscr{D}^*(\mathcal{E}) \to \mathscr{D}(\mathcal{E})$ is fully faithful for any $* \in \{+, -, b\}$. If $\mathcal{E}$ is closed under cokernels of monomorphisms in $\mathcal{A}$, then a complex $X^\bullet \in \mathscr{C}^b(\mathcal{E})$ is strictly exact if and only if it is exact in $\mathscr{C}(\mathcal{A})$.

For more details on derived categories of exact categories, we refer the reader to [Keller 1996]. The following result follows from [Keller 1996, Theorem 12.1].

Lemma 3.2. *Let $\mathcal{E}$ be a full subcategory of an abelian category $\mathcal{A}$, satisfying the two conditions*:

(i) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$ with $X \in \mathcal{E}$, then $Y \in \mathcal{E}$ if and only if $Z \in \mathcal{E}$.*

(ii) *For each object $M \in \mathcal{A}$, there is a long exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ in $\mathcal{A}$ for a natural number n such that $E_i \in \mathcal{E}$ for all $0 \le i \le n$.*

Then the inclusion $\mathcal{E} \subseteq \mathcal{A}$ induces a triangle equivalence $\mathscr{D}^+(\mathcal{E}) \xrightarrow{\simeq} \mathscr{D}^+(\mathcal{A})$ which can be restricted to an equivalence $\mathscr{D}^b(\mathcal{E}) \xrightarrow{\simeq} \mathscr{D}^b(\mathcal{A})$.

Remark 3.3. If (ii) in Lemma 3.2 is strengthened as

(ii′) There is a natural number n such that, for each object $M \in \mathcal{A}$, there is a long exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

in $\mathcal{A}$ with $E_i \in \mathcal{E}$ for all $0 \le i \le n$,

then the inclusion $\mathcal{E} \subseteq \mathcal{A}$ induces a triangle equivalence $\mathscr{D}(\mathcal{E}) \to \mathscr{D}(\mathcal{A})$ which restricts to an equivalence $\mathscr{D}^*(\mathcal{E}) \to \mathscr{D}^*(\mathcal{A})$ for any $* \in \{+, -, b\}$.

For a proof of this result and its dual statement, we refer the reader to [Positselski 2017, Proposition A.5.6]

Proof of Theorem 1.2(1). It follows from (a), (b) and Lemma 2.3 that $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$ and that both $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $\mathcal{A}$. In particular, (D2) in Definition 1.1 holds.

Now, we keep all the notation introduced in Section 2.3. Under the assumptions of (a), (b) and (c), we show that the functor $\mathscr{D}^*(i) : \mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A})$, induced from the inclusion $i : \mathcal{X} \to \mathcal{A}$, is fully faithful.

By Lemma 2.5(1), $\mathcal{A}_{r\text{-}adj}$ is closed under extensions in $\mathcal{A}$ and thus a fully exact subcategory of $\mathcal{A}$. Moreover, i has a right adjoint $r : \mathcal{A} \to \mathcal{X}$ which is an exact functor when restricted to $\mathcal{A}_{r\text{-}adj}$ by Lemma 2.5(3). Thus $r : \mathcal{A}_{r\text{-}adj} \to \mathcal{X}$ induces a derived functor $\mathscr{D}^*(r) : \mathscr{D}^*(\mathcal{A}_{r\text{-}adj}) \to \mathscr{D}^*(\mathcal{X})$. Since the functor i is fully faithful, the composition of i and r is isomorphic to the identity functor of $\mathcal{X}$. Thus $(\mathscr{D}^*(i), \mathscr{D}^*(r))$ is an adjoint pair and the composition of $\mathscr{D}^*(i)$ with $\mathscr{D}^*(r)$ is isomorphic to the identity functor of $\mathscr{D}^*(\mathcal{X})$. This implies that $\mathscr{D}^*(i) : \mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A}_{r\text{-}adj})$ is fully faithful. Further, for each object $M \in \mathcal{A}$, it follows from (c) that there is a monomorphism $M \to I$ in $\mathcal{A}$ such that $I \in \mathcal{A}_{r\text{-}adj}$. Since $\mathcal{A}_{r\text{-}adj}$ is closed under quotients in $\mathcal{A}$ by Lemma 2.5(1), there is an exact sequence $0 \to M \to I \to J \to 0$ in $\mathcal{A}$ with $I, J \in \mathcal{A}_{r\text{-}adj}$. By Remark 3.3, the inclusion $\mathcal{A}_{r\text{-}adj} \subseteq \mathcal{A}$ of exact categories induces a triangle equivalence $\mathscr{D}^*(\mathcal{A}_{r\text{-}adj}) \xrightarrow{\simeq} \mathscr{D}^*(\mathcal{A})$ for any $* \in \{b, +, -, \varnothing\}$. Consequently, $\mathscr{D}^*(i) : \mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A})$ is fully faithful.

Dually, under (a), (b) and (d), the functor $\mathscr{D}^*(j) : \mathscr{D}^*(\mathcal{Y}) \to \mathscr{D}^*(\mathcal{A})$, induced from the inclusion $j : \mathcal{Y} \to \mathcal{A}$, is fully faithful. This follows from Lemmas 2.5(2) and 2.5(4), and the dual statement of Lemma 3.2.

Thus, Definition 1.1(D1) is satisfied. We identify $\mathscr{D}^*(\mathcal{X})$ and $\mathscr{D}^*(\mathcal{Y})$ with $\mathrm{Im}(\mathscr{D}^*(i))$ and $\mathrm{Im}(\mathscr{D}^*(j))$, respectively. It remains to check Definition 1.1(D3).

Let $N^\bullet \in \mathscr{D}^*(\mathcal{A})$. Since $\mathscr{D}^*(\mathcal{A}_{r\text{-}adj}) \xrightarrow{\simeq} \mathscr{D}^*(\mathcal{A})$, there is a complex $M^\bullet := (M^i)_{i \in \mathbb{Z}} \in \mathscr{C}^*(\mathcal{A}_{r\text{-}adj})$ such that $N^\bullet \simeq M^\bullet$ in $\mathscr{D}^*(\mathcal{A})$. Moreover, since $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$, it follows from Lemma 2.4(2) that each morphism $f : M \to N$ in $\mathcal{A}$ extends uniquely to a morphism $\varepsilon_f : \varepsilon_M \to \varepsilon_N$ of exact sequences (see Lemma 2.4 for notation). Applying this to the differentials of $M^\bullet$

yields a long exact sequence

$$\varepsilon_{M^\bullet}: \qquad 0 \longrightarrow Y_{M^\bullet} \xrightarrow{\varepsilon^{-2}_{M^\bullet}} X_{M^\bullet} \xrightarrow{\varepsilon^{-1}_{M^\bullet}} M^\bullet \xrightarrow{\varepsilon^{0}_{M^\bullet}} Y^{M^\bullet} \xrightarrow{\varepsilon^{1}_{M^\bullet}} X^{M^\bullet} \longrightarrow 0$$

in $\mathscr{C}^*(\mathcal{A})$ such that $X_{M^\bullet}, X^{M^\bullet} \in \mathscr{C}^*(\mathcal{X})$ and $Y_{M^\bullet}, Y^{M^\bullet} \in \mathscr{C}^*(\mathcal{Y})$. Since $M^i \in \mathcal{A}_{r\text{-}adj}$ for any $i \in \mathbb{Z}$, we have $X^{M^i} = 0$. This implies $X^{M^\bullet} = 0$ and thus the exact sequence $\varepsilon_{M^\bullet}$ is of the form:

$$\varepsilon_{M^\bullet}: \qquad 0 \longrightarrow Y_{M^\bullet} \xrightarrow{\varepsilon^{-2}_{M^\bullet}} X_{M^\bullet} \xrightarrow{\varepsilon^{-1}_{M^\bullet}} M^\bullet \xrightarrow{\varepsilon^{0}_{M^\bullet}} Y^{M^\bullet} \longrightarrow 0.$$

Let $Z_{M^\bullet}$ be the mapping cone of the chain map $\varepsilon^{-1}_{M^\bullet}$. Then $X_{M^\bullet} \to M^\bullet \to Z_{M^\bullet} \to X_{M^\bullet}[1]$ is a distinguished triangle in $\mathscr{D}^*(\mathcal{A})$. We show $Z_{M^\bullet} \in \mathscr{D}^*(\mathcal{Y})$.

Since $\varepsilon^{-1}_{M^\bullet}$ is the composite of the surjection $X_{M^\bullet} \to \mathrm{Coker}(\varepsilon^{-2}_{M^\bullet})$ with the injection $\mathrm{Coker}(\varepsilon^{-2}_{M^\bullet}) \to M^\bullet$, there is a triangle $Y_{M^\bullet}[1] \to Z_{M^\bullet} \to Y^{M^\bullet} \to Y_{M^\bullet}[2]$ in $\mathscr{D}^*(\mathcal{A})$ (constructed from the octahedral axiom of triangulated categories). As $\mathscr{D}^*(\mathcal{Y})$ is a full triangulated subcategory of $\mathscr{D}^*(\mathcal{A})$, it follows from $Y_{M^\bullet}, Y^{M^\bullet} \in \mathscr{D}^*(\mathcal{Y})$ that $Z_{M^\bullet} \in \mathscr{D}^*(\mathcal{Y})$.

Since $N^\bullet \simeq M^\bullet$ in $\mathscr{D}^*(\mathcal{A})$, there is a triangle $X_{M^\bullet} \to N^\bullet \to Z_{M^\bullet} \to X_{M^\bullet}[1]$ satisfying that $X_{M^\bullet} \in \mathscr{D}^*(\mathcal{X})$ and $Z_{M^\bullet} \in \mathscr{D}^*(\mathcal{Y})$. This shows Definition 1.1(D3). Thus $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathscr{D}^*(\mathcal{A})$. □

As a consequence of Theorem 1.2(1), we have

Corollary 3.4. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\mathcal{A}$ with $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$. If for each object $M \in \mathcal{A}$, there is a short exact sequence $0 \to X \to M \to Y \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, then $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$ for any $* \in \{b, +, -, \varnothing\}$.*

3.2. ***Necessary conditions for the existence of derived decompositions.*** Throughout this section, $\mathcal{A}$ is an abelian category and $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$. We keep all the notation in Section 2.3.

Lemma 3.5. *Let $M \in \mathcal{A}$ and $N \in \mathcal{Y}$.*

(1) *There is a long exact sequence of extension groups for $n \in \mathbb{Z}$:*

$$\cdots \longrightarrow \mathrm{Ext}^{n-2}_{\mathcal{A}}(Y_M, N) \longrightarrow \mathrm{Ext}^{n}_{\mathcal{A}}(\ell(M), N) \longrightarrow \mathrm{Ext}^{n}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Ext}^{n-1}_{\mathcal{A}}(Y_M, N) \longrightarrow \cdots$$

(2) *If $M \in {}^{\perp 1}\mathcal{Y}$, then $\ell(M) \in {}^{\perp 1}\mathcal{Y}$.*

(3) *If $M \in {}^{\perp >0}\mathcal{Y}$, then $\mathrm{Ext}^{n-2}_{\mathcal{A}}(Y_M, N) \simeq \mathrm{Ext}^{n}_{\mathcal{A}}(\ell(M), N)$ for all $n \geq 2$.*

Proof. (1) We define $C = \mathrm{Im}(\varepsilon^{-1}_M)$ and $K = \mathrm{Im}(\varepsilon^{0}_M)$ in ε_M. Then we have two exact sequences $0 \to Y_M \to r(M) \to C \to 0$ and $0 \to K \to \ell(M) \to X^M \to 0$. It follows from the first exact sequence that $\mathrm{Ext}^{n-1}_{\mathcal{A}}(Y_M, N) \simeq \mathrm{Ext}^{n}_{\mathcal{A}}(C, N)$ for all

$n \in \mathbb{Z}$. Similarly, it follows from the second exact sequence that $\mathrm{Ext}^n_{\mathcal{A}}(\ell(M), N) \simeq \mathrm{Ext}^n_{\mathcal{A}}(K, N)$ for all $n \in \mathbb{Z}$. We then apply $\mathrm{Ext}^n_{\mathcal{A}}(-, N)$ to the exact sequence $0 \to C \to M \to K \to 0$ and get the desired exact sequence. Note that (2) and (3) follow from (1). □

From Lemma 3.5, we have the following

Corollary 3.6. *Assume that $\mathcal{A}$ has enough projectives. Let $P \in \mathcal{P}(\mathcal{A})$ and $N \in \mathcal{Y}$. Then*

(1) $\ell(P) \in {}^{\perp 1}\mathcal{Y}$ *and* $\mathrm{Ext}^{n-2}_{\mathcal{A}}(Y_P, N) \simeq \mathrm{Ext}^n_{\mathcal{A}}(\ell(P), N)$ *for all* $n \geq 2$.

(2) $\ell(P) \in {}^{\perp >0}\mathcal{Y}$ *if and only if* $Y_P = 0$.

Proof. If $P \in \mathcal{P}(\mathcal{A})$, then $P \in {}^{\perp >0}\mathcal{Y}$. Now, (1) follows from Lemma 3.5(2)–(3). Further, by (1), $\ell(P) \in {}^{\perp >0}\mathcal{Y}$ if and only if $\mathrm{Ext}^m_{\mathcal{A}}(Y_P, Z) = 0$ for all $Z \in \mathcal{Y}$ and for all $m \geq 0$. Due to $Y_P \in \mathcal{Y}$, $\mathrm{Hom}_{\mathcal{A}}(Y_P, Y_P) = 0$ implies $Y_P = 0$. Thus (2) holds. □

We need the following result from [Geigle and Lenzing 1991, Proposition 4.3].

Lemma 3.7. *Let $\mathcal{B}$ be an abelian full subcategory of $\mathcal{A}$ and let $\lambda : \mathcal{B} \to \mathcal{A}$ be the inclusion. Then $\mathcal{D}^b(\lambda) : \mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A})$ is fully faithful if and only if, for any $X, Y \in \mathcal{B}$ and for any $n \in \mathbb{N}$, the homomorphism $\varphi^n_{X,Y} : \mathrm{Ext}^n_{\mathcal{B}}(X, Y) \to \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ induced from $\mathcal{D}^b(\lambda)$ is an isomorphism.*

The following result, which will be used in Section 4, is implied by Lemmas 3.7 and 3.5(1). Here, we omit its proof.

Corollary 3.8. (1) *Suppose that $\mathcal{D}^b(j) : \mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(\mathcal{A})$ is fully faithful. If $I \in \mathcal{I}(\mathcal{Y})$, then $\mathrm{Ext}^n_{\mathcal{A}}(M, I) = 0$ for any $M \in \mathcal{A}$ and $n \geq 2$.*

(2) *Suppose that $\mathcal{D}^b(i) : \mathcal{D}^b(\mathcal{X}) \to \mathcal{D}^b(\mathcal{A})$ is fully faithful. If $P \in \mathcal{P}(\mathcal{X})$, then $\mathrm{Ext}^n_{\mathcal{A}}(P, M) = 0$ for any $M \in \mathcal{A}$ and $n \geq 2$.*

Lemma 3.9. *Let $\mathcal{B}$ be an abelian full subcategory of $\mathcal{A}$.*

(1) *Assume that $\mathcal{B}$ has enough projectives. Then the derived functor $\mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A})$ induced from the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is fully faithful if and only if $\mathcal{P}(\mathcal{B}) = \mathcal{B} \cap {}^{\perp >0}\mathcal{B}$.*

(2) *Assume that $\mathcal{B}$ has enough injectives. Then the derived functor $\mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A})$ induced from the inclusion $\mathcal{B} \subseteq \mathcal{A}$ is fully faithful if and only if $\mathcal{I}(\mathcal{B}) = \mathcal{B} \cap \mathcal{B}^{\perp >0}$.*

Proof. We only prove (1) since (2) can be proved dually. Note that the inclusions $\mathcal{B} \cap {}^{\perp >0}\mathcal{B} \subseteq \mathcal{B} \cap {}^{\perp 1}\mathcal{B} \subseteq \mathcal{P}(\mathcal{B})$ always hold.

Let $\lambda : \mathcal{B} \to \mathcal{A}$ be the inclusion. By Lemma 3.7, to prove (1), it is enough to show that, for any $X, Y \in \mathcal{B}$ and for any $n \in \mathbb{N}$, the homomorphism $\varphi^n_{X,Y} : \mathrm{Ext}^n_{\mathcal{B}}(X, Y) \to \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is an isomorphism. Clearly, $\varphi^0_{X,Y}$ is the identity map. So it suffices to check that $\varphi^n_{X,Y}$ is an isomorphism for $n \geq 1$.

Suppose that $\mathscr{D}^b(\lambda)$ is fully faithful. If $X \in \mathscr{P}(\mathcal{B})$, then $\mathrm{Ext}^n_{\mathcal{B}}(X, Y) = 0$ for all $n \geq 1$, and therefore $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$. Thus $X \in {}^{\perp>0}\mathcal{B}$ and $\mathscr{P}(B) \subseteq \mathcal{B} \cap {}^{\perp>0}\mathcal{B}$. So the necessity of Lemma 3.9(1) holds.

Conversely, suppose $\mathscr{P}(B) = \mathcal{B} \cap {}^{\perp>0}\mathcal{B}$. This implies that $\varphi^n_{X,Y}$ is an isomorphism for $X \in \mathscr{P}(\mathcal{B})$ because $\mathrm{Ext}^n_{\mathcal{B}}(X, Y) = 0 = \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$. Since $\mathcal{B}$ has enough projectives, any object $X \in \mathcal{B}$ has a projective resolution in $\mathcal{B}$. This resolution is also exact in $\mathcal{A}$ because $\mathcal{B}$ is an abelian full subcategory of $\mathcal{A}$. Now, for a fixed object $Y \in \mathcal{B}$, we apply the functors $\mathrm{Ext}^i_{\mathcal{B}}(-, Y)$ and $\mathrm{Ext}^i_{\mathcal{A}}(-, Y)$ for $i \in \mathbb{N}$ to this resolution and then get two long exact sequences of extension groups, linked by commutative diagrams. Note that $\mathrm{Ext}^i_{\mathcal{A}}(P, Y) = 0$ for all $i \geq 1$ and $P \in \mathscr{P}(\mathcal{B})$. By induction on n and by the five lemma, we can show that $\varphi^n_{X,Y}$ are isomorphisms for $n \geq 1$. Thus $\mathscr{D}^b(\lambda)$ is fully faithful. □

Lemma 3.10. (1) *If $\mathcal{A}$ has enough projectives, then so does $\mathcal{Y}$, and $\mathscr{P}(\mathcal{Y}) = \mathrm{add}(\{\ell(P) \mid P \in \mathscr{P}(\mathcal{A})\})$.*

(2) *If $\mathcal{A}$ has enough injectives, then so does $\mathcal{X}$, and $\mathscr{I}(\mathcal{X}) = \mathrm{add}(\{r(I) \mid I \in \mathscr{I}(\mathcal{A})\})$.*

Proof. (1) Since j is an exact functor and (ℓ, j) is an adjoint pair, ℓ is a right exact functor and preserves projective objects. This means $\ell(P) \in \mathscr{P}(\mathcal{Y})$ for $P \in \mathscr{P}(\mathcal{A})$. Given any object $Y \in \mathcal{Y}$, since $\mathcal{A}$ has enough projectives, there exists an epimorphism $\pi : Q \to j(Y)$ in $\mathcal{A}$ with $Q \in \mathscr{P}(\mathcal{A})$. Hence $\ell(\pi) : \ell(Q) \to \ell(j(Y))$ is an epimorphism in $\mathcal{Y}$. As $\ell(j(Y)) \simeq Y$, $\ell(\pi)$ is an epimorphism from $\ell(Q)$ to Y. This shows that $\mathcal{Y}$ has enough projectives. Moreover, if $Y \in \mathscr{P}(\mathcal{Y})$, then Y is a direct summand of $\ell(Q)$. This shows (1). (2) can be proved dually. □

The next result characterizes when $\mathscr{D}^b(i)$ and $\mathscr{D}^b(j)$ are fully faithful.

Lemma 3.11. (1) *If $\mathcal{A}$ has enough projectives, then $\mathscr{D}^b(j) : \mathscr{D}^b(\mathcal{Y}) \to \mathscr{D}^b(\mathcal{A})$ is fully faithful if and only if $Y_P = 0$ for any object $P \in \mathscr{P}(\mathcal{A})$.*

(2) *If $\mathcal{A}$ has enough injectives, then $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$ is fully faithful if and only if $X^I = 0$ for any object $I \in \mathscr{I}(\mathcal{A})$.*

Proof. We show (1) by Lemma 3.9(1). By Lemmas 3.10(1) and 3.9(1), the functor $\mathscr{D}^b(j)$ is fully faithful if and only if $\ell(P) \in {}^{\perp>0}\mathcal{Y}$ for all $P \in \mathscr{P}(\mathcal{A})$. But the latter is equivalent to saying $Y_P = 0$ by Corollary 3.6(2). Thus (1) holds. Dually, (2) can be proved by Lemma 3.9(2). □

Proof of Theorem 1.2(2). The sufficiency is a direct consequence of Theorem 1.2(1). To show the necessity, we suppose that $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$. Then (a) and (b) hold by Proposition 3.1. If $\mathcal{A}$ has enough injectives, then (c) and (c′) are equivalent. Similarly, if $\mathcal{A}$ has enough projectives, then (d) and (d′) are equivalent. Note that $\mathscr{D}^*(i) : \mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A})$ can be restricted to bounded derived categories. By Definition 1.1(D1), $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$ is fully faithful. Similarly, $\mathscr{D}^b(j) :$

$\mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(\mathcal{A})$ is fully faithful. Since $\mathcal{A}$ has enough projectives and injectives, it follows from Lemma 3.11 that (c′) and (d′) in Theorem 1.2(2) hold. □

3.3. ***Derived decompositions versus semiorthogonal decompositions.*** In this section we establish relations between derived decompositions of abelian categories and semiorthogonal decompositions of several derived categories.

The following result is an easy observation from Definitions 2.1 and 1.1.

Lemma 3.12. *Let $\mathcal{A}$ be an abelian category, $\mathcal{X}$ and $\mathcal{Y}$ abelian subcategories of $\mathcal{A}$ and $* \in \{b, +, -, \varnothing\}$. Suppose that the inclusions $i : \mathcal{X} \subseteq \mathcal{A}$ and $j : \mathcal{Y} \subseteq \mathcal{A}$ induce fully faithful functors $\mathcal{D}^*(i) : \mathcal{D}^*(\mathcal{X}) \to \mathcal{D}^*(\mathcal{A})$ and $\mathcal{D}^*(j) : \mathcal{D}^*(\mathcal{Y}) \to \mathcal{D}^*(\mathcal{A})$, respectively. If $(\mathrm{Im}(\mathcal{D}^*(i)), \mathrm{Im}(\mathcal{D}^*(j)))$ is a semiorthogonal decomposition of $\mathcal{D}^*(\mathcal{A})$, then $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{D}^*$-decomposition of $\mathcal{A}$.*

To obtain the converse of Lemma 3.12, we consider abelian categories with additional properties.

Definition 3.13. Let $\mathcal{A}$ be an abelian category.

(1) $\mathcal{A}$ is complete (respectively, cocomplete) if products (respectively, coproducts) indexed over sets exist in $\mathcal{A}$; and bicomplete if it is complete and cocomplete.

(2) $\mathcal{A}$ satisfies AB4 if it is cocomplete such that coproducts of short exact sequences (indexed over sets) are exact. Dually, $\mathcal{A}$ satisfies AB4′ if it is complete and products of short exact sequences (indexed over sets) are exact.

If $\mathcal{A}$ satisfies AB4, then $\mathcal{D}(\mathcal{A})$ has coproducts indexed over sets, and therefore the coproducts of distinguished triangles in $\mathcal{D}(\mathcal{A})$ are distinguished triangles. Moreover, $\mathcal{D}(\mathcal{A})$ itself is the smallest full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ containing $\mathcal{A}$ and being closed under coproducts. Dually, if $\mathcal{A}$ satisfies AB4′, then $\mathcal{D}(\mathcal{A})$ has products indexed over sets, and therefore the products of triangles in $\mathcal{D}(\mathcal{A})$ are again triangles.

Note that if a cocomplete abelian category has enough injectives, then it satisfies AB4. Dually, if a complete abelian category has enough projectives, then it satisfies AB4′. Examples of abelian categories with both AB4 and AB4′ are the module categories of rings, and the categories of additive functors from essentially small triangulated categories to the category of abelian groups (see [Neeman 2001, Chapter 6] for details).

Lemma 3.14. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{X}$ and $\mathcal{Y}$ be abelian subcategories of $\mathcal{A}$. Assume that*

(1) *$\mathcal{X}$ satisfies* AB4 *and the inclusion $i : \mathcal{X} \subseteq \mathcal{A}$ preserves coproducts,*

(2) *$\mathcal{Y}$ satisfies* AB4′ *and the inclusion $j : \mathcal{Y} \subseteq \mathcal{A}$ preserves products, and*

(3) $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[n]) = 0$ *for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $n \in \mathbb{Z}$.*

Then $\operatorname{Hom}_{\mathscr{D}(\mathcal{A})}(i(X^\bullet), j(Y^\bullet)) = 0$ *for all* $X^\bullet \in \mathscr{D}(\mathcal{X})$ *and* $Y^\bullet \in \mathscr{D}(\mathcal{Y})$.

Proof. For any $Y \in \mathcal{Y}$, let $\mathscr{X}(Y)$ be the full subcategory of $\mathscr{D}(\mathcal{X})$ consisting of objects $X^\bullet$ such that $\operatorname{Hom}_{\mathscr{D}(\mathcal{A})}(i(X^\bullet), j(Y)[n]) = 0$ for all $n \in \mathbb{Z}$. Then $\mathscr{X}(Y)$ is a full triangulated subcategory of $\mathscr{D}(\mathcal{X})$. By (1), $\mathscr{D}(\mathcal{X})$ has coproducts and $\mathscr{X}(Y) \subseteq \mathscr{D}(\mathcal{X})$ is closed under coproducts. Moreover, $\mathcal{X} \subseteq \mathscr{X}(Y)$ by (3). Note that $\mathscr{D}(\mathcal{X})$ is the smallest full triangulated subcategory of $\mathscr{D}(\mathcal{X})$ containing $\mathcal{X}$ and being closed under coproducts. Thus $\mathscr{X}(Y) = \mathscr{D}(\mathcal{X})$. It follows that $\operatorname{Hom}_{\mathscr{D}(\mathcal{A})}(i(X^\bullet), j(Y)[n]) = 0$ for all $X^\bullet \in \mathscr{D}(\mathcal{X})$ and $n \in \mathbb{Z}$. Dually, when $X^\bullet \in \mathscr{D}(\mathcal{X})$ is fixed, one can apply (2) and (3) to prove $\operatorname{Hom}_{\mathscr{D}(\mathcal{A})}(i(X^\bullet), j(Y^\bullet)) = 0$ for all $Y^\bullet \in \mathscr{D}(\mathcal{Y})$. □

Proposition 3.15. *Suppose that* $\mathcal{A}$ *is an abelian category satisfying* AB4 *and* AB4′. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be abelian subcategories of* $\mathcal{A}$ *and let* $* \in \{b, +, -, \varnothing\}$. *Then the following are equivalent*:

(1) $(\mathcal{X}, \mathcal{Y})$ *is a* $\mathscr{D}^*$*-decomposition of* $\mathcal{A}$.

(2) *The inclusions* $i : \mathcal{X} \subseteq \mathcal{A}$ *and* $j : \mathcal{Y} \subseteq \mathcal{A}$ *induce fully faithful functors* $\mathscr{D}^*(i) : \mathscr{D}^*(\mathcal{X}) \to \mathscr{D}^*(\mathcal{A})$ *and* $\mathscr{D}^*(j) : \mathscr{D}^*(\mathcal{Y}) \to \mathscr{D}^*(\mathcal{A})$, *respectively*, *and* $(\operatorname{Im}(\mathscr{D}^*(i)), \operatorname{Im}(\mathscr{D}^*(j)))$ *is a semiorthogonal decomposition of* $\mathscr{D}^*(\mathcal{A})$.

Proof. (2) implies (1) by Lemma 3.12. Suppose (1) holds. Then $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$ by Proposition 3.1. In particular, $\mathcal{X} = {}^{\perp}\mathcal{Y}$. It follows that $\mathcal{X}$ is closed under coproducts in $\mathcal{A}$ and $i : \mathcal{X} \subseteq \mathcal{A}$ preserves coproducts. Since $\mathcal{A}$ satisfies AB4, $\mathcal{X}$ also satisfies AB4. Dually, $\mathcal{Y}$ is closed under products in $\mathcal{A}$ and satisfies AB4′. Now, (2) holds by Definitions 1.1 and 2.1 and Lemma 3.14. □

As a consequence of Theorem 1.2(2) and Proposition 3.15, we can construct half recollements of derived categories from derived decompositions.

Corollary 3.16. *Let* $\mathcal{A}$ *be a bicomplete abelian category with enough projectives and injectives. If* $(\mathcal{X}, \mathcal{Y})$ *is a derived decomposition of* $\mathcal{A}$, *then there exists a lower half recollement*

$$\mathscr{D}^*(\mathcal{X}) \underset{\mathbb{R}^*(r)}{\overset{\mathscr{D}^*(i)}{\rightleftarrows}} \mathscr{D}^*(\mathcal{A}) \underset{\mathscr{D}^*(j)}{\overset{\mathbb{L}^*(\ell)}{\rightleftarrows}} \mathscr{D}^*(\mathcal{Y})$$

for $* \in \{b, +, -, \varnothing\}$, *where* $\mathbb{R}^*(r)$ *and* $\mathbb{L}^*(\ell)$ *denote the right- and left-derived functors of* r *and* ℓ, *respectively.*

Remark 3.17. Consider the statements:

(1) $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}$-decomposition of $\mathcal{A}$;

(2) $(\mathcal{X}, \mathcal{Y})$ is a $\mathscr{D}^*$-decomposition of $\mathcal{A}$ for any $* \in \{+, -\}$;

(3) $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Thus the existence of $\mathscr{D}^b$-decompositions is the weakest condition among those other type of derived decompositions introduced in Definition 1.1. This is why we sometimes pay more attention to the existence of such decompositions.

To show the above, we consider the triangle given in Definition 1.1(D3). If $H^n(M^\bullet) = H^{n+1}(M^\bullet) = 0$ for some integer n, then $H^{n+1}(X_{M^\bullet}) \simeq H^n(Y^{M^\bullet})$. Clearly, $H^{n+1}(X_{M^\bullet}) \in \mathcal{X}$ and $H^n(Y^{M^\bullet}) \in \mathcal{Y}$ since $\mathcal{X}$ and $\mathcal{Y}$ are abelian subcategories of $\mathcal{A}$. However, $\mathcal{X} \cap \mathcal{Y} = \{0\}$ by Definition 1.1(D2). Thus $H^{n+1}(X_{M^\bullet}) = H^n(Y^{M^\bullet}) = 0$.

Question. Does (3) always imply (1)? Theorem 1.2(2) tells us this is the case if $\mathcal{A}$ has enough projectives and injectives.

4. Constructing derived decompositions of module categories

In this section we first apply Theorem 1.2 to show that homological ring epimorphisms can provide derived decompositions (see Proposition 4.1), and then prove that localizing subcategories and right perpendicular subcategories in abelian categories also give rise to derived decompositions (see Proposition 4.6). Finally, we construct derived decompositions for module categories over left nonsingular rings and commutative noetherian rings (see Corollaries 1.5 and 4.10, respectively). This construction shows also that the module category of a commutative ring with the Krull dimension at most 1 admits a derived stratification (see Corollary 4.11).

4.1. ***Homological ring epimorphisms*** **.** In this section we show that homological ring epimorphisms produce not only derived decompositions, but also derived equivalences and recollements.

Throughout this section, we assume that $\lambda : R \to S$ is a homological ring epimorphism. Define

$$\mathcal{A} := R\text{-Mod}, \quad \mathcal{X} := S\text{-Mod},$$
$$\mathcal{Y} := \{Y \in R\text{-Mod} \mid \mathrm{Hom}_R(S, Y) = 0 = \mathrm{Ext}^1_R(S, Y)\},$$
$$\mathcal{Z} := \{Z \in R\text{-Mod} \mid S \otimes_R Z = 0 = \mathrm{Tor}^R_1(S, Z)\}.$$

Proposition 4.1. (1) *$(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$ if and only if* $\mathrm{projdim}({}_RS) \leq 1$.

(2) *$(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$ if and only if* $\mathrm{projdim}({}_RS) \leq 1$ *and* $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), \mathrm{Ker}(\lambda)) = 0$.

Proof. (1) Since λ is a ring epimorphism, the restriction functor $\lambda_* : \mathcal{X} \to \mathcal{A}$ is fully faithful. So, we identify $\mathcal{X}$ with the image of λ_*. Further, since λ is homological, the derived functor $\mathscr{D}^b(\lambda_*) : \mathscr{D}^b(S) \to \mathscr{D}^b(R)$ is fully faithful. Note that ${}_SS \in \mathscr{P}(\mathcal{X})$, the category of projective S-modules. If $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$, then $\mathrm{projdim}({}_RS) \leq 1$ by Corollary 3.8(2). This shows the necessity of (1).

To show the sufficiency of (1), we assume $\mathrm{projdim}({}_RS) \leq 1$. Then $\mathcal{Y} = S^{\perp}$. It follows from [Geigle and Lenzing 1991, Proposition 1.1] that $\mathcal{Y}$ is an abelian full subcategory of $\mathcal{A}$. Since ${}^{\perp}\mathcal{Y}$ contains ${}_RS$ and is closed under direct sums in $\mathcal{A}$, it must contain all projective S-modules. Moreover, each object of $\mathcal{X}$ admits a projective resolution by projective S-modules. Consequently, for any $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $n \in \mathbb{N}$, $\mathrm{Hom}_{\mathscr{D}^b(R)}(X, Y[n]) \simeq \mathrm{Hom}_{\mathscr{D}^b(R)}(\Omega_S^n(X), Y) = 0$, where $\Omega_S^n(X)$ denotes an n-th syzygy module of ${}_SX$. This implies $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$. By Lemma 2.3(2), to show (1), it suffices to prove that $(\mathcal{X}, \mathcal{Y})$ satisfies (GC).

The functor $\mathscr{D}^b(\lambda_*) : \mathscr{D}^b(S) \to \mathscr{D}^b(R)$ has a right adjoint functor $\mathbb{R}\mathrm{Hom}_R(S, -) : \mathscr{D}^b(R) \to \mathscr{D}^b(S)$. Let $\varepsilon : \mathscr{D}^b(\lambda_*)\ \mathbb{R}\mathrm{Hom}_R(S, -) \to \mathrm{Id}_{\mathscr{D}^b(R)}$ be the counit adjunction. Then, for each R-module M, there exists a distinguished triangle in $\mathscr{D}(R)$:

$$(\dagger) \qquad \mathscr{D}^b(\lambda_*)\,\mathbb{R}\mathrm{Hom}_R(S, M) \xrightarrow{\varepsilon_M} M \longrightarrow Y_M^{\bullet} \longrightarrow \mathscr{D}^b(\lambda_*)\,\mathbb{R}\mathrm{Hom}_R(S, M)[1].$$

Since λ is homological, $\mathscr{D}^b(\lambda_*)$ is fully faithful. So $\mathbb{R}\mathrm{Hom}_R(S, \varepsilon_M)$ is an isomorphism in $\mathscr{D}^b(S)$. This means $Y_M^{\bullet} \in \mathscr{Y} := \mathrm{Ker}(\mathbb{R}\mathrm{Hom}_R(S, -)) \subseteq \mathscr{D}^b(R)$. Thus

$$Y_M^{\bullet} \in \mathscr{Y} = \{Y^{\bullet} \in \mathscr{D}^b(R) \mid \mathrm{Hom}_{\mathscr{D}^b(R)}(S, Y^{\bullet}[n]) = 0 \text{ for all } n \in \mathbb{Z}\}$$

and

$$\mathcal{Y} = \mathscr{Y} \cap R\text{-Mod}.$$

Taking cohomologies on the triangle (†) yields an exact sequence of R-modules:

$$0 \longrightarrow H^{-1}(Y_M^{\bullet}) \longrightarrow \mathrm{Hom}_R(S, M) \xrightarrow{\mathrm{Hom}_R(\lambda, M)} M \longrightarrow H^0(Y_M^{\bullet}) \longrightarrow \mathrm{Ext}_R^1(S, M) \longrightarrow 0,$$

where M is identified with $\mathrm{Hom}_R(R, M)$. Clearly, $\mathrm{Hom}_R(S, M)$ and $\mathrm{Ext}_R^1(S, M)$ belong to $\mathcal{X}$. On the other hand, since $\mathrm{projdim}({}_RS) \leq 1$, the R-module S is isomorphic in $\mathscr{D}^b(R)$ to a two-term complex of projective R-modules and there is the exact sequence by [Chen and Xi 2012, Lemma 3.4]:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}^b(R)}(S, H^{n-1}(Y^{\bullet})[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^b(R)}(S, Y^{\bullet}[n]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^b(R)}(S, H^n(Y^{\bullet})) \longrightarrow 0.$$

This shows $\mathscr{Y} = \{Y^{\bullet} \in \mathscr{D}^b(R) \mid H^n(Y^{\bullet}) \in \mathcal{Y} \text{ for all } n \in \mathbb{Z}\}$. It then follows from $Y_M^{\bullet} \in \mathscr{Y}$ that $H^i(Y_M^{\bullet}) \in \mathcal{Y}$ for any $i \in \mathbb{Z}$. Now, we define $X_M := \mathrm{Hom}_R(S, M)$, $X^M := \mathrm{Ext}_R^1(S, M)$, $Y_M := H^{-1}(Y_M^{\bullet})$ and $Y^M := H^0(Y_M^{\bullet})$. This shows the sufficiency of (1).

(2) Clearly, $\mathcal{A}$ has enough projectives and injectives. If M is injective, then $X^M = 0$. Note that $Y_M \simeq \mathrm{Ker}(\mathrm{Hom}_R(\lambda, M)) \simeq \mathrm{Hom}_R(\mathrm{Coker}(\lambda), M)$ as R-modules. By

(1) and Theorem 1.2, $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$ if and only if $\mathrm{projdim}({}_R S) \leq 1$ and $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), M) = 0$ whenever M is projective. To check $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), M) = 0$ for projective modules M, we only need to show $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), R) = 0$. Because $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), -)$ commutes with products and each projective R-module can be embedded into a product of copies of R. However, since λ is a ring epimorphism, $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), S) = 0$. This implies $\mathrm{Hom}_R(\mathrm{Coker}(\lambda), R) \simeq \mathrm{Hom}_R(\mathrm{Coker}(\lambda), \mathrm{Ker}(\lambda))$. Thus (2) holds. □

When dealing with flat dimensions, we have

Proposition 4.2. (1) $(\mathcal{Z}, \mathcal{X})$ *is a complete Ext-orthogonal pair in* $\mathcal{A}$ *if and only if* $\mathrm{flatdim}(S_R) \leq 1$.

(2) $(\mathcal{Z}, \mathcal{X})$ *is a derived decomposition of* $\mathcal{A}$ *if and only if* $\mathrm{flatdim}(S_R) \leq 1$ *and* $\mathrm{Coker}(\lambda) \otimes_R I = 0$ *for any injective* R*-module* I.

Proof. The proof of this result is similar to the one of Proposition 4.1. For the convenience of the reader, we only sketch a few key points of the proof.

Let $J := \mathrm{Hom}_{\mathbb{Z}}(S_S, \mathbb{Q}/\mathbb{Z})$. Then J is an injective cogenerator in S-Mod. Further, $\mathrm{injdim}({}_R J) = \mathrm{flatdim}(S_R)$ because a right R-module N is flat if and only if ${}_R\,\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ is injective. So the necessity of (1) follows from Corollary 3.8(1). If $\mathrm{flatdim}(S_R) \leq 1$, then

(1) $\mathcal{Z}$ is abelian full subcategory of $\mathcal{A}$.

(2) For each $M^\bullet \in \mathscr{D}(R)$, $S \otimes^{\mathbb{L}}_R M^\bullet = 0$ if and only if $H^n(M^\bullet) \in \mathcal{Z}$ for all $n \in \mathbb{Z}$.

(3) For each $M \in \mathcal{A}$, there is an exact sequence $0 \to \mathrm{Tor}_1^R(S, M) \to X_M \to M \to S \otimes_R M \to X^M \to 0$ of R-modules such that $X_M, X^M \in \mathcal{Z}$. Clearly, $X^M \simeq \mathrm{Coker}(\lambda) \otimes_R M$ as R-modules. Now, all other assertions in Proposition 4.2 can be concluded from Theorem 1.2. □

As a consequence of Proposition 4.2, we have the result on localizations of commutative rings.

Corollary 4.3. *Let* R *be a commutative noetherian ring*, Φ *a multiplicative subset of* R, S *the localization of* R *at* Φ *and* $\mathcal{U} := \{X \in R\text{-Mod} \mid S \otimes_R X = 0\}$. *Then* $(\mathcal{U}, S\text{-Mod})$ *is a derived decomposition of* R-Mod.

Proof. Let $\lambda : R \to S$ be the localization of R at Φ. Then S is commutative and flat as an R-module, and therefore λ is a homological ring epimorphism. By Proposition 4.2(2), it suffices to show $\mathrm{Coker}(\lambda) \otimes_R I = 0$ (or equivalently, $\lambda \otimes_R I$ is surjective) for any injective R-module I.

Since R is a commutative noetherian ring, each injective R-module is a direct sum of indecomposable injective R-modules (see [Enochs and Jenda 2000, Theorem 3.3.10]). So we only need to check the surjection of $\lambda \otimes_R I$ whenever I is indecomposable. By [Enochs and Jenda 2000, Theorem 3.3.7], there is a prime ideal

$\mathfrak{p}$ of R such that I is isomorphic to the injective envelope $E(R/\mathfrak{p})$ of the R-module $R/\mathfrak{p}$. Moreover, by [Enochs and Jenda 2000, Theorem 3.3.8(6)], $\lambda \otimes_R E(R/\mathfrak{p})$ is an isomorphism if $\Phi \cap \mathfrak{p} = \varnothing$; and $S \otimes_R E(R/\mathfrak{p}) = 0$ if $\Phi \cap \mathfrak{p} \neq \varnothing$. This implies that $\lambda \otimes_R E(R/\mathfrak{p})$ is always surjective, and therefore $\lambda \otimes_R I$ is surjective. Thus Corollary 4.3 follows from Proposition 4.2(2). □

Next, we show that the derived decompositions in Propositions 4.1 and 4.2 provide also lower half recollements of derived categories.

For the ring epimorphism $\lambda : R \to S$, we get naturally a complex $Q^\bullet : 0 \to R \xrightarrow{\lambda} S \to 0$ of R-R-bimodules with R and S in degrees -1 and 0, respectively. Let

$$F := Q^\bullet[-1] \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(R), \quad G := \mathbb{R}\mathrm{Hom}_R(Q^\bullet[-1], -) : \mathscr{D}(R) \to \mathscr{D}(R)$$

and let $\mathrm{Tria}({}_R Q^\bullet)$ be the smallest full triangulated subcategory of $\mathscr{D}(R)$ containing $Q^\bullet$ and being closed under direct sums. Then (F, G) is an adjoint pair of triangle functors and the restriction of G to $\mathrm{Tria}({}_R Q^\bullet)$ is fully faithful (see [Nicolás and Saorín 2009, §4]).

In the case (2) of Proposition 4.1, it follows from Corollary 3.16 that there is a lower half recollement of derived categories:

$$(\ddagger) \qquad \mathscr{D}^*(S) \underset{\mathbb{R}^* \mathrm{Hom}_R(S,-)}{\overset{\mathscr{D}^*(\lambda_*)}{\rightleftarrows}} \mathscr{D}^*(R) \underset{\mathscr{D}^*(j)}{\overset{\mathbb{L}^*(\ell)}{\rightleftarrows}} \mathscr{D}^*(\mathcal{Y})$$

for $* \in \{b, +, -, \varnothing\}$, where $\ell : R\text{-Mod} \to \mathcal{Y}$ is a left adjoint of the inclusion $j : \mathcal{Y} \to R$-Mod. We claim that ℓ is the composition of the functors:

$$R\text{-Mod} \hookrightarrow \mathscr{D}^*(R) \xrightarrow{G} \mathscr{D}^*(R) \xrightarrow{H^0} R\text{-Mod}.$$

In fact, there is a canonical triangle $Q^\bullet[-1] \xrightarrow{\sigma} R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^\bullet$ in $\mathscr{D}^b(R \otimes_{\mathbb{Z}} R^{\mathrm{op}})$ which induces a sequence of triangle functors from $\mathscr{D}^*(R)$ to $\mathscr{D}^*(R)$

$$(\ddagger) \qquad G[-1] \xrightarrow{\pi_*} \mathscr{D}(\lambda_*)\, \mathbb{R}\mathrm{Hom}_R(S, -) \xrightarrow{\lambda_*} \mathbb{R}\mathrm{Hom}_R(R, -) \xrightarrow{\sigma_*} G$$

such that their operations on a fixed object in $\mathscr{D}^*(R)$ yield a triangle in $\mathscr{D}^*(R)$. Clearly, $\mathbb{R}\mathrm{Hom}_R(R, -)$ can be identified with the identity functor of $\mathscr{D}^*(R)$ up to natural isomorphism. So, for an R-module M, by taking cohomologies on (‡), we get a long exact sequence of R-modules:

$$0 \longrightarrow H^{-1}(G(M)) \longrightarrow \mathrm{Hom}_R(S, M) \xrightarrow{\mathrm{Hom}_R(\lambda, M)} M \longrightarrow H^0(G(M)) \longrightarrow \mathrm{Ext}^1_R(S, M) \longrightarrow 0.$$

Now, as in the proof of Proposition 4.1(1), both $\mathrm{Hom}_R(S, M)$ and $\mathrm{Ext}^1_R(S, M)$ belong to $\mathcal{X}$ and both $H^{-1}(G(M))$ and $H^0(G(M))$ belong to $\mathcal{Y}$. Thus $\ell(M) = H^0(G(M))$ by the definition of ℓ.

Similarly, in the case (2) of Proposition 4.2, we obtain a lower half recollement:

$$\mathscr{D}^*(\mathcal{Z}) \underset{\mathbb{R}^*(r)}{\overset{\mathscr{D}^*(i)}{\rightleftarrows}} \mathscr{D}^*(R) \underset{\mathscr{D}^*(\lambda_*)}{\overset{S\otimes^{\mathbb{L}}_R -}{\rightleftarrows}} \mathscr{D}^*(S) \ ,$$

where $r := H^0F(-) : R\text{-Mod} \to \mathcal{Z}$ is a right adjoint of the inclusion $i : \mathcal{Z} \to R\text{-Mod}$. The five-term exact sequence of R-modules is given by $0 \to \mathrm{Tor}^R_1(S, M) \to H^0(F(M)) \to M \to S \otimes_R M \to H^1(F(M)) \to 0$.

Proof of Corollary 1.3. This follows from Propositions 4.1(2) and 4.2(2) together with the above-mentioned two half-recollements. □

An example of Corollary 1.3(3) reads as follows: Let R be a 1-Gorenstein ring (that is, a commutative noetherian ring R with $\mathrm{injdim}(R) \leq 1$) and let Φ be the set of all nonzero divisors of R. Then λ is always injective, and ${}_RS$ is flat, injective and of projective dimension at most 1, satisfying (a) and (b) of Theorem 1.2. In the case $\mathbb{Z} \subseteq \mathbb{Q}$, we get a recollement $(\mathscr{D}^*(\mathbb{Q}), \mathscr{D}^*(\mathbb{Z}), \mathscr{D}^*(\mathcal{Y}))$ and an equivalence $\mathscr{D}^*(\mathcal{Y}) \simeq \mathscr{D}^*(\mathcal{Z})$.

4.2. ***Localizing subcategories.*** In this section we construct derived decompositions from localizing subcategories.

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a full subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ is a *Serre* subcategory if it is closed under subobjects, quotients and extensions. In particular, $\mathcal{X}$ is an abelian subcategory of $\mathcal{A}$, and the *quotient category* $\mathcal{A}/\mathcal{X}$ (in the sense of Gabriel, Grothendieck and Serre) is defined by inverting all these morphisms in $\mathcal{A}$ that have kernels and cokernels in $\mathcal{X}$. The quotient category has the same objects as $\mathcal{A}$ and is again an abelian category. Moreover, there is a canonical exact functor $q : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ (called the quotient functor) such that the kernel of q is exactly $\mathcal{X}$.

A Serre subcategory $\mathcal{X}$ of $\mathcal{A}$ is called a *localizing* subcategory of $\mathcal{A}$ if q has a right adjoint $s : \mathcal{A}/\mathcal{X} \to \mathcal{A}$ (called the section functor). This is equivalent to saying that q restricts to an equivalence of additive categories from $\mathcal{X}^{\perp 0,1} := \mathcal{X}^{\perp 0} \cap \mathcal{X}^{\perp 1}$ to $\mathcal{A}/\mathcal{X}$ (see [Gabriel 1962, Chapter III.2; Geigle and Lenzing 1991, Proposition 2.2]). In this case, $\mathcal{X} = {}^{\perp 0,1}(\mathcal{X}^{\perp 0,1})$. Note that $\mathcal{X}^{\perp 0,1}$ is closed under extensions and kernels in $\mathcal{A}$ (see, for example, [Geigle and Lenzing 1991, Proposition 1.1]), but it may not be an abelian subcategory of $\mathcal{A}$ in general.

If $\mathcal{A}$ is a Grothendieck category (that is, an abelian category with a generator and coproducts such that direct limits of exact sequences are exact), then a Serre subcategory of $\mathcal{A}$ is localizing if and only if it is closed under coproducts in $\mathcal{A}$ (see [Geigle and Lenzing 1991, Proposition 2.5]).

Lemma 4.4. *Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a localizing subcategory of $\mathcal{A}$ with $\mathcal{Y} := \mathcal{X}^{\perp}$. Then*

(1) *$(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$ if and only if $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$ if and only if the section functor $s : \mathcal{A}/\mathcal{X} \to \mathcal{A}$ is exact.*

(2) *$(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$ if and only if both $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$ and the derived functor $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$, induced from the inclusion $i : \mathcal{X} \to \mathcal{A}$, is fully faithful.*

Proof. (1) Since $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$, it follows from [Geigle and Lenzing 1991, Proposition 2.2] that, for each object $M \in \mathcal{A}$, there is an exact sequence $0 \to X_1 \to M \to \overline{M} \to X_2 \to 0$ in $\mathcal{A}$ with $X_1, X_2 \in \mathcal{X}$ and $\overline{M} \in \mathcal{X}^{\perp_{0,1}}$. Clearly, $\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}$. By Lemma 2.3, $(\mathcal{X}, \mathcal{Y})$ is a complete Ext-orthogonal pair in $\mathcal{A}$ if and only if $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$. It remains to show that $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$ if and only if s is exact.

Let $\mathcal{B} := \mathcal{A}/\mathcal{X}$ and $q_1 : \mathcal{X}^{\perp_{0,1}} \xrightarrow{\simeq} \mathcal{B}$ the restriction of the canonical functor $q : \mathcal{A} \to \mathcal{B}$ to $\mathcal{X}^{\perp_{0,1}}$. It is known that s is always fully faithful and isomorphic to the composition of the quasi-inverse of q_1 with the inclusion $\mathcal{X}^{\perp_{0,1}} \subseteq \mathcal{A}$ (see, for example, [Geigle and Lenzing 1991, Proposition 2.2]). If $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$, then $\mathcal{Y}$ is an abelian subcategory of $\mathcal{A}$ since $\mathcal{X}^{\perp_{0,1}}$ is closed under extensions and kernels in $\mathcal{A}$. In this case, q_1 is an equivalence of abelian categories, and thus s is exact.

Conversely, suppose that s is an exact functor. Since both q and s are exact, they induce derived functors $\mathscr{D}^b(q) : \mathscr{D}^b(\mathcal{A}) \to \mathscr{D}^b(\mathcal{B})$ and $\mathscr{D}^b(s) : \mathscr{D}^b(\mathcal{B}) \to \mathscr{D}^b(\mathcal{A})$ such that $(\mathscr{D}^b(q), \mathscr{D}^b(s))$ is an adjoint pair and $\mathscr{D}^b(s)$ is fully faithful. Picking up an object $Y \in \mathcal{X}^{\perp_{0,1}}$, we then have $Y \simeq s(Z)$ for some $Z \in \mathcal{B}$ since $\mathrm{Im}(s) = \mathcal{X}^{\perp_{0,1}}$. For any $X \in \mathcal{X}$ and $n \in \mathbb{N}$, there are

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \simeq \mathrm{Ext}^n_{\mathcal{A}}(X, s(Z)) &= \mathrm{Hom}_{\mathscr{D}^b(\mathcal{A})}(X, s(Z)[n]) \\ &\simeq \mathrm{Hom}_{\mathscr{D}^b(\mathcal{B})}(q(X), Z[n]) = \mathrm{Hom}_{\mathscr{D}^b(\mathcal{B})}(0, Z[n]) = 0.\end{aligned}$$

This implies both $\mathcal{X}^{\perp_{0,1}} \subseteq \mathcal{Y}$ and $\mathcal{X}^{\perp_{0,1}} = \mathcal{Y}$.

(2) The necessity of the conditions in (2) is a consequence of Definition 1.1, Proposition 3.1 and (1). Now, we show the sufficiency of the conditions in (2).

Suppose that $\mathscr{D}^b(i)$ is fully faithful and $\mathcal{Y} = \mathcal{X}^{\perp_{0,1}}$. By the proof of (1), $(\mathscr{D}^b(q), \mathscr{D}^b(s))$ is an adjoint pair and $\mathscr{D}^b(s)$ is fully faithful. Let $\mathscr{X} := \mathrm{Ker}(\mathscr{D}^b(q))$. Then $(\mathscr{X}, \mathrm{Im}(\mathscr{D}^b(s)))$ is a semiorthogonal decomposition of $\mathscr{D}^b(\mathcal{A})$. Since q is exact, $\mathscr{X}$ coincides with the full triangulated subcategory of $\mathscr{D}^b(\mathcal{A})$ consisting of complexes $X^\bullet \in \mathscr{D}(\mathcal{A})$ such that $H^n(X^\bullet) \in \mathcal{X}$ for all n. Further, since $\mathcal{X}$ is an abelian subcategory of $\mathcal{A}$ and $\mathscr{D}^b(i)$ is fully faithful, we have $\mathscr{X} = \mathrm{Im}(\mathscr{D}^b(i))$. Recall that s is fully faithful and $\mathrm{Im}(s) = \mathcal{Y}$. Thus $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$. $\square$

Lemma 4.5. *Suppose that $\mathcal{A}$ is an abelian category such that each object of $\mathcal{A}$ has an injective envelope. Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{A}$ and $\mathcal{E} := \mathcal{X}^{\perp 0} \cap \mathcal{I}(\mathcal{A})$. Then $\mathcal{X}^{\perp}$ (respectively, $\mathcal{X}^{\perp 0,1}$) consists of all objects M which has a minimal injective resolution $0 \to M \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ with $I_i \in \mathcal{E}$ for all $i \geq 0$ (respectively, $i = 0, 1$).*

Proof. We first prove that $\mathcal{X}^{\perp 0}$ is closed under injective envelope in $\mathcal{A}$, that is, if $Z \in \mathcal{X}^{\perp 0}$, then the injective envelope $E(Z)$ of Z belongs to $\mathcal{X}^{\perp 0}$.

Let $Z \in \mathcal{X}^{\perp 0}$ and assume contrarily that there is a nonzero morphism $f : X \to E(Z)$ in $\mathcal{A}$ for some $X \in \mathcal{X}$. Then $\mathrm{Im}(f) \neq 0$ and there is a monomorphism $g : \mathrm{Im}(f) \to E(Z)$. Let $h : Z \to E(Z)$ be an injective envelope of Z. Taking the pull-back of (g, h) yields another two monomorphisms $K \to Z$ and $K \to \mathrm{Im}(f)$ in $\mathcal{A}$. As $E(Z)$ is the injective envelope of Z, we have $K \neq 0$. By assumption, $\mathcal{X}$ is closed under subobjects and quotients. Hence, with X also $\mathrm{Im}(f)$ and K lie in $\mathcal{X}$. It follows from $Z \in \mathcal{X}^{\perp 0}$ that $K = 0$, a contradiction. This shows $E(Z) \in \mathcal{X}^{\perp 0}$. Hence $\mathcal{X}^{\perp 0}$ is closed under injective envelope in $\mathcal{A}$.

If $Z \in \mathcal{X}^{\perp 0}$, then $E(Z) \in \mathcal{E}$. Moreover, there are inclusions of categories: $\mathcal{E} \subseteq \mathcal{X}^{\perp} \subseteq \mathcal{X}^{\perp 0,1} \subseteq \mathcal{A}$. Recall that $\mathcal{X}^{\perp}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $\mathcal{A}$, and that $\mathcal{X}^{\perp 0,1}$ is closed under extensions and kernels in $\mathcal{A}$. Now, it is easy to verify Lemma 4.5. $\square$

The following result furnishes a way to get derived decompositions from localizing subcategories.

Proposition 4.6. *Let $\mathcal{A}$ be an abelian category such that each of its objects has an injective envelope. If $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$ with $\mathcal{Y} := \mathcal{X}^{\perp}$, then the following are equivalent*:

(1) *$(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of $\mathcal{A}$.*

(2) *Each morphism $I^0 \to I^1$ between injective objects in $\mathcal{A}$ with $I^1 \in \mathcal{Y}$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ such that I^2 is injective and $I^2 \in \mathcal{Y}$.*

(3) *The image of each morphism from an injective object in $\mathcal{A}$ to an object in $\mathcal{Y}$ belongs to $\mathcal{Y}$.*

Proof. Since $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$, the proof of Lemma 4.4(1) shows that the five-term exact sequence associated with an object $M \in \mathcal{A}$ becomes

$$0 \longrightarrow r(M) \xrightarrow{\varepsilon_M^{-1}} M \xrightarrow{\varepsilon_M^{0}} \ell(M) \longrightarrow X^M \longrightarrow 0$$

with $r(M)$, $X^M \in \mathcal{X}$ and $\ell(M) \in \mathcal{X}^{\perp 0,1}$.

(1) ⇒ (2) Since $\mathcal{A}$ has enough injectives and $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$ is fully faithful, we see from Lemma 3.11(2) that $X^M = 0$ whenever $M \in \mathscr{I}(\mathcal{A})$. Let $f : I^0 \to I^1$ be a morphism of injective objects in $\mathcal{A}$ with $I^1 \in \mathcal{Y}$. Then there is an exact sequence $0 \to r(I^0) \to I^0 \to \ell(I^0) \to 0$ in $\mathcal{A}$. We always have $\mathrm{Hom}_{\mathcal{A}}(r(I^0), I^1) = 0$, due to $r(I^0) \in \mathcal{X}$ and $I^1 \in \mathcal{Y}$. Consequently, f is the composition of the morphism $\varepsilon^0_{I^0} : I^0 \to \ell(I^0)$ with another morphism $g : \ell(I^0) \to I^1$. This implies $\mathrm{Coker}(f) \simeq \mathrm{Coker}(g)$. Since $\mathcal{Y}$ is an abelian subcategory of $\mathcal{A}$, $\mathrm{Coker}(g) \in \mathcal{Y}$, and thus also $\mathrm{Coker}(f) \in \mathcal{Y}$. Let I^2 be the injective envelope of $\mathrm{Coker}(f)$. Then $I^2 \in \mathcal{Y}$ by Lemma 4.5. Now, we extend f to an exact sequence $I^0 \to I^1 \to I^2$.

(2) ⇒ (3) Thanks to Lemma 4.5, the assumption (2) implies $\mathcal{Y} = \mathcal{X}^{\perp 0,1}$. Since $\mathcal{Y}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $\mathcal{A}$ and since $\mathcal{X}^{\perp 0,1}$ is closed under kernels in $\mathcal{A}$ by [Geigle and Lenzing 1991, Proposition 1.1], $\mathcal{Y}$ is an abelian subcategory of $\mathcal{A}$. Let $h : I^0 \to Y$ be morphism in $\mathcal{A}$ with $Y \in \mathcal{Y}$ and I^0 an injective object. Further, let I^1 be the injective envelope of Y with a monomorphism $s : Y \to I^1$. Then $I^1 \in \mathcal{Y}$, according to Lemma 4.5. Moreover, by (2), the composition h with s can be completed to an exact sequence $I^0 \xrightarrow{hs} I^1 \xrightarrow{t} I^2$ in $\mathcal{A}$ such that I^2 is injective and $I^2 \in \mathcal{Y}$. Thus $\mathrm{Im}(hs) = \mathrm{Ker}(t) \in \mathcal{Y}$. Since $\mathrm{Im}(h) \simeq \mathrm{Im}(hs)$, $\mathrm{Im}(h) \in \mathcal{Y}$, and therefore (3) follows.

(3) ⇒ (1) By Lemma 4.5, $\mathcal{Y} = \mathcal{X}^{\perp 0,1}$. Thus, by Lemma 4.4(2), to show (1), it suffices to prove that $\mathscr{D}^b(i) : \mathscr{D}^b(\mathcal{X}) \to \mathscr{D}^b(\mathcal{A})$ is fully faithful.

In fact, given an injective object I of $\mathcal{A}$, since $\ell(I) \in \mathcal{Y}$, (3) implies that the image of $\varepsilon^0_I : I \to \ell(I)$ belongs to $\mathcal{Y}$. Further, since $\varepsilon^0_I : I \to \ell(I)$ is the unit adjunction of I, $\mathrm{Im}(\varepsilon^0_I) = \ell(I)$. This shows $X^I = 0$. Now, by Lemma 3.11, $\mathscr{D}^b(i)$ is fully faithful since $\mathcal{A}$ has enough injectives. Thus (1) holds. □

4.3. ***Nonsingular rings.*** In this section we will apply Proposition 4.6 to construct derived decomposition from left nonsingular rings. The main result of this section is Corollary 1.5.

First, we recall some notions on left nonsingular rings (see [Goodearl 1976, Chapter 1]).

Let R be a ring and M be an R-module with a submodule N. Recall that M is an *essential extension* of N (or N is an *essential submodule* of M) if every nonzero submodule of M has nonzero intersection with N. Recall that the injective envelope of N is just an essential extension M of N with M an injective module. As before, M is denoted by $E(N)$. The set of all essential submodules of ${}_RR$ is denoted by $\mathscr{S}(R)$. A class $\mathcal{U}$ of R-modules is said to be *closed under essential extensions* in R-Mod provided that $M \in \mathcal{U}$ whenever M is an essential extension of a module $N \in \mathcal{U}$.

For an R-module M, we define $Z(M) := \{x \in M \mid Ix = 0 \text{ for some } I \in \mathcal{S}(R)\}$. This is a submodule of M and called the *singular submodule* of M. The module M is called a *singular* module if $Z(M) = M$; and a *nonsingular* module if $Z(M) = 0$. The ring R is said to be *left nonsingular* if ${}_RR$ is a nonsingular module. Examples of left nonsingular rings include left semihereditary rings, direct products of integral domains, semiprime left Goldie rings and commutative semiprime rings (see [Goodearl 1976] for more examples).

To show Corollary 1.5, we need the following basic properties of singular and nonsingular modules (see [Goodearl 1976, Propositions 1.20 and 1.22] for proofs).

Lemma 4.7. *Let R be a ring, $\mathcal{X}$ the full subcategory of singular modules in R-*Mod*, and M an R-module.*

(1) *$M \in \mathcal{X}$ if and only if M is isomorphic to the quotient X/Y of an essential extension $Y \subseteq X$.*

(2) *$M \in \mathcal{X}^{\perp 0}$ if and only if M is nonsingular.*

(3) *$\mathcal{X}$ is closed under submodules, quotients and direct sums in R-*Mod; *and $\mathcal{X}^{\perp 0}$ is closed under submodules, direct products, extensions and essential extensions in R-*Mod.

In general, the full subcategory $\mathcal{X}$ of singular R-modules may not be closed under extensions in R-Mod. Nevertheless, the next lemma, taken from [Goodearl 1976, Propositions 1.23 and 2.12], provides a positive situation.

Lemma 4.8. *Let R be a left nonsingular ring and $\mathcal{X}$ be the full subcategory of singular R-modules. Then*

(1) *$\mathcal{X}$ is closed under extensions and essential extensions in R-*Mod.

(2) $\mathcal{X} = {}^{\perp 0}(\mathcal{X}^{\perp 0}) = {}^{\perp 0}E(R)$.

(3) *Let M be an R-module. Then $M/Z(M) \in \mathcal{X}^{\perp 0}$. Moreover, M is nonsingular if and only if M can be embedded in a direct product of copies of $E(R)$.*

Remark 4.9. If R is a left nonsingular ring, then the subcategory $\mathcal{X}$ of singular R-modules is a localizing subcategory of R-Mod by Lemmas 4.7 and 4.8. Hence $(\mathcal{X}, \mathcal{X}^{\perp 0})$ is a hereditary torsion pair in R-Mod (see [Beligiannis and Reiten 2007, §1, p. 13] for definition). Moreover, $E(R)$ is nonsingular, while $E(R)/R$ is singular.

Proof of Corollary 1.5. Let R be a left nonsingular ring. We show $\mathcal{Y} = \mathcal{X}^{\perp 0} \cap \mathcal{I}(R\text{-Mod}) = \mathcal{X}^{\perp} = \mathcal{X}^{\perp 0,1}$.

Since $E({}_RR)$ is injective and nonsingular, $\mathcal{Y} \subseteq \mathcal{X}^{\perp 0} \cap \mathcal{I}(R\text{-Mod})$. The converse inclusion $\mathcal{X}^{\perp 0} \cap \mathcal{I}(R\text{-Mod}) \subseteq \mathcal{Y}$ follows from Lemmas 4.7(2) and 4.8(3). Thus $\mathcal{Y} = \mathcal{X}^{\perp 0} \cap \mathcal{I}(R\text{-Mod})$.

Clearly, $\mathcal{Y} \subseteq \mathcal{X}^{\perp} \subseteq \mathcal{X}^{\perp 0,1}$. So, to show the other equalities, it is enough to show $\mathcal{X}^{\perp 0,1} \subseteq \mathcal{Y}$. To this purpose, we first prove a general result:

(∗∗) If $0 \to M \xrightarrow{f} I \xrightarrow{g} J$ is an exact sequence of R-module such that I is injective and J is nonsingular, then M is injective.

In fact, since f is injective and ${}_RI$ is injective, there is another injective R-module K such that $\mathrm{Coker}(f) \simeq (E(M)/M) \oplus K$. Moreover, since J is nonsingular, $\mathrm{Im}(g)$ is also nonsingular by Lemma 4.7(3). It follows from $\mathrm{Coker}(f) = \mathrm{Im}(g)$ that $E(M)/M$ is nonsingular. However, $E(M)/M$ is singular by Lemma 4.7(1). This implies that $M = E(M)$ is injective.

By Lemma 4.5, the category $\mathcal{X}^{\perp 0,1}$ consists of all R-modules Y which has a minimal injective presentation $0 \to Y \to I_0 \to I_1$ with $I_0, I_1 \in \mathcal{Y}$. Since I_0 is injective and I_1 is nonsingular, it follows from (∗∗) that Y is injective. Thus $Y \in \mathrm{add}(I_0) \subseteq \mathcal{Y}$. This shows $\mathcal{X}^{\perp 0,1} \subseteq \mathcal{Y}$.

As a consequence of (∗∗), there is an isomorphism $I \simeq M \oplus \mathrm{Im}(g)$. Thus $\mathrm{Im}(g)$ is injective. This implies $\mathrm{Im}(g) \in \mathrm{add}(J)$. Now, if $J \in \mathcal{Y}$, then $\mathrm{Im}(g) \in \mathcal{Y}$. Hence Corollary 1.5 follows from Proposition 4.6(3). □

4.4. ***Commutative noetherian rings.*** In this section we will apply Proposition 4.6 to construct derived decomposition from commutative noetherian rings. Thus we prove Corollary 1.4 in the introduction.

Let R be a commutative noetherian ring and $\mathrm{Spec}(R)$ be the prime spectrum of R. For a multiplicative subset Σ of R, we denote by $\Sigma^{-1}R$ the localization of R at Σ. For $\mathfrak{p} \in \mathrm{Spec}(R)$, let $R_\mathfrak{p}$ be the localization of R at the set $R \setminus \mathfrak{p}$. We always identify $\mathrm{Spec}(\Sigma^{-1}R)$ with the subset of all prime ideals $\mathfrak{p}$ of R satisfying $\mathfrak{p} \cap \Sigma = \varnothing$. We also regard $(\Sigma^{-1}R)$-Mod as a full subcategory of R-Mod in the sense that an R-module M belongs to $(\Sigma^{-1}R)$-Mod if and only if $M \simeq (\Sigma^{-1}R) \otimes_R M$ as R-modules. Let $\mathrm{Ass}(M)$ be the set of prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is isomorphic to a submodule of M. Thus $\mathrm{Ass}(M) = \mathrm{Ass}(E(M))$, where $E(M)$ is an injective envelope of M. The *support* of M, denoted by $\mathrm{Supp}(M)$, is by definition the set of prime ideals $\mathfrak{p}$ of R satisfying $\mathrm{Tor}_i^R(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},\ M) \neq 0$ for some $i \in \mathbb{N}$ (see [Foxby 1979]). In general, $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$. The second inclusion is an equality if the module M is finitely generated. Note that $\mathrm{Supp}(M)$ is the union of the subsets $\mathrm{Ass}(I)$ of $\mathrm{Spec}(R)$, where I runs over all those injective R-modules that appear in a minimal injective resolution of M (see [Foxby 1979, Remark 2.9] or [Krause 2008, Lemma 3.3]). In particular, if M is injective, then $\mathrm{Ass}(M) = \mathrm{Supp}(M)$.

The following hold for a commutative noetherian ring R:

(a) Each injective R-module is a direct sum of indecomposable injective R-modules.

(b) $\{E(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$ is a complete set of nonisomorphic indecomposable injective R-modules.

(c) $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q}) \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$ in $\mathrm{Spec}(R)$ (see [Enochs and Jenda 2000, Theorems 3.3.7 and 3.3.8]).

Let $\mathcal{S}$ be a full subcategory of R-Mod and let Φ be a subset of $\mathrm{Spec}(R)$. We define

$$\mathrm{Supp}(\mathcal{S}) := \bigcup_{M\in\mathcal{S}} \mathrm{Supp}(M) \quad \text{and} \quad \mathrm{Supp}^{-1}(\Phi) := \{M \in R\text{-Mod} \mid \mathrm{Supp}(M) \subseteq \Phi\}.$$

Gabriel's [1962, p. 425] classification of localizing subcategories conveys that the map Supp induces a bijection between the set of localizing subcategories of R-Mod and the set of specialization closed subsets of $\mathrm{Spec}(R)$. The inverse of Supp is just given by Supp^{-1}. This was extended in [Krause 2008, Theorem 3.1] to a bijection (with the same maps) between the set of abelian full subcategories of R-Mod closed under extensions and arbitrary direct sums, and the set of coherent subsets of $\mathrm{Spec}(R)$.

A subset Φ of $\mathrm{Spec}(R)$ is said to be *specialization closed* provided that if $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in \Phi$ implies $\mathfrak{q} \in \Phi$; *coherent* provided that each homomorphism $I^0 \to I^1$ between injective R-modules with $\mathrm{Ass}(I^0) \cup \mathrm{Ass}(I^1) \subseteq \Phi$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ such that I^2 is injective and $\mathrm{Ass}(I^2) \subseteq \Phi$ (see [Krause 2008, §3]). Examples of coherent subsets are specialization closed subsets and $\mathrm{Spec}(\Sigma^{-1}R)$. For further information on coherent subsets, we refer the reader to [Krause 2008, §4].

An application of Proposition 4.6 is the following

Corollary 4.10. *Let R be a commutative noetherian ring, Φ be a specialization closed subset of* $\mathrm{Spec}(R)$, *and*

$$\Phi^c := \mathrm{Spec}(R) \setminus \Phi.$$

Then the pair $(\mathrm{Supp}^{-1}(\Phi), \mathrm{Supp}^{-1}(\Phi^c))$ *is a derived decomposition of* R-Mod *if and only if Φ^c is coherent.*

Proof. Let $\mathcal{X} := \mathrm{Supp}^{-1}(\Phi)$ and $\mathrm{Ass}^{-1}(\Phi^c) := \{M \in R\text{-Mod} \mid \mathrm{Ass}(M) \subseteq \Phi^c\}$. We first show that $\mathcal{X}^{\perp 0} = \mathrm{Ass}^{-1}(\Phi^c)$.

Let $U \in \mathcal{X}$ and $V \in \mathrm{Ass}^{-1}(\Phi^c)$. Then $\mathrm{Supp}(E(U)) \subseteq \mathrm{Supp}(U) \subseteq \Phi$ and $\mathrm{Ass}(E(V)) = \mathrm{Ass}(V) \subseteq \Phi^c$. If $\mathrm{Hom}_R(E(U), E(V)) \neq 0$, then there is a nonzero homomorphism from a direct summand $E(R/\mathfrak{p})$ of $E(U)$ to a direct summand $E(R/\mathfrak{q})$ of $E(V)$, where $\mathfrak{p} \in \Phi$ and $\mathfrak{q} \in \Phi^c$. In this case, we have $\mathfrak{p} \subseteq \mathfrak{q}$. This is contradictory to the assumption that Φ is specialization closed. Thus $\mathrm{Hom}_R(E(U), E(V)) = 0$. This implies $\mathrm{Hom}_R(U, V) = 0$ and shows $\mathrm{Ass}^{-1}(\Phi^c) \subseteq \mathcal{X}^{\perp 0}$. To verify $\mathcal{X}^{\perp 0} \subseteq \mathrm{Ass}^{-1}(\Phi^c)$, we take $W \in \mathcal{X}^{\perp 0}$ and $\mathfrak{a} \in \mathrm{Ass}(W)$. Then $R/\mathfrak{a}$ is isomorphic to a nonzero submodule of W. If $\mathfrak{a} \in \Phi$ and Φ is specialization closed, then $\mathrm{Supp}(R/\mathfrak{a}) = \{\mathfrak{b} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{b}\} \subseteq \Phi$. This implies $R/\mathfrak{a} \in \mathcal{X}$ and $\mathrm{Hom}_R(R/\mathfrak{a}, W) = 0$, a contradiction. Thus $\mathfrak{a} \in \Phi^c$.

Let $\mathcal{Y} := \mathcal{X}^{\perp}$ and $\mathcal{E} := \mathcal{X}^{\perp_0} \cap \mathcal{I}(R\text{-Mod})$. Then $\mathcal{E} = \mathrm{Ass}^{-1}(\Phi^c) \cap \mathcal{I}(R\text{-Mod}) = \mathcal{Y} \cap \mathcal{I}(R\text{-Mod})$. By Lemma 4.5, $\mathcal{Y}$ consists of all R-modules M which has a minimal injective resolution $0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$ such that $I_i \in \mathcal{E}$ for all $i \geq 0$. Since $\mathrm{Supp}(M) = \bigcup_{i\geq 0} \mathrm{Ass}(I_i)$, $\mathcal{Y} = \mathrm{Supp}^{-1}(\Phi^c)$. Observe that, for any $\Psi \subseteq \mathrm{Spec}(R)$, $\mathrm{Supp}^{-1}(\Psi)$ is always closed under direct sums in R-Mod because $\mathrm{Supp}\big(\bigoplus_{j\in J} M_j\big) = \bigcup_{j\in J} \mathrm{Supp}(M_j)$ for any family $\{M_j\}_{j\in J}$ of R-modules with J an index set. In particular, $\mathcal{Y}$ is closed under direct sums in R-Mod.

If $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of R-Mod, then $\mathcal{Y}$ is an abelian full subcategory of R-Mod and closed under both extensions and direct sums, and therefore Φ^c is coherent.

Conversely, suppose that Φ^c is coherent. Let $f : I^0 \to I^1$ be a homomorphism between injective R-modules with $I^1 \in \mathcal{Y}$. By Proposition 4.6(2), we need to extend f to an exact sequence $I^0 \to I^1 \to I^2$ in R-Mod with $I^2 \in \mathcal{E}$. This can be done if $I^0 \in \mathcal{E}$ since Φ^c is coherent. For the general case, we decompose I^0 into a direct sum of indecomposable injective modules. Recall that $\{E(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$ is a complete set of isomorphism classes of indecomposable injective R-modules and that

$$\mathrm{Ass}(E(R/\mathfrak{p})) = \mathrm{Supp}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}.$$

Consequently, $E(R/\mathfrak{p})$ belongs to either $\mathcal{X}$ or $\mathcal{Y}$. This yields a decomposition $I^0 = X \oplus Y$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Since $\mathrm{Hom}_R(X, I^1) = 0$, $f = (0, g)$, where $g : Y \to I^1$ is the restriction of f to Y. Clearly, g is a homomorphism between modules in $\mathcal{E}$. Now, we first extend g and then f to an exact sequence $I^0 \to I^1 \to I^2$ in R-Mod with $I^2 \in \mathcal{E}$. Thus $(\mathcal{X}, \mathcal{Y})$ is a derived decomposition of R-Mod. □

Proof of Corollary 1.4. For (1), the statement (i) is Corollary 4.10. For (ii), if the Krull dimension of R is at most 1, then every subset of $\mathrm{Spec}(R)$ is coherent by [Krause 2008, Theorem 1.2]. Thus (ii) follows from (i). For (2), let S be the localization of R at Σ and recall that $\mathrm{Spec}(S)$ is identified with the subset of all prime ideals $\mathfrak{p}$ of R satisfying $\mathfrak{p} \cap \Sigma = \varnothing$. Then $\Phi^c = \mathrm{Spec}(S)$ is coherent and (2) follows from (i). Note that (2) follows also from Corollary 4.3. □

In Corollary 4.10, when Φ^c is coherent, $\mathcal{Y} := \mathrm{Supp}^{-1}(\Phi^c)$ is an abelian full subcategory of R-Mod closed under direct sums, and the inclusion $j : \mathcal{Y} \to R$-Mod has a left adjoint $\ell : R\text{-Mod} \to \mathcal{Y}$. Let $S = \mathrm{End}_R(\ell(R))$ and let $\lambda : R \to S$ be the ring homomorphism induced from ℓ. By [Geigle and Lenzing 1991, Proposition 3.8], λ is a ring epimorphism and induces an equivalence of abelian categories: $S\text{-Mod} \xrightarrow{\simeq} \mathcal{Y}$. Moreover, S is a flat R-module since ℓ is exact. Thus λ is a flat ring epimorphism (see also [Angeleri Hügel et al. 2020]). Consequently, S is also a commutative noetherian ring. So we can apply Corollary 4.10 (for example, via localizations) to S and obtain a derived decomposition of S-Mod. By iterating this procedure, we can stratify R-Mod as a sequence of derived decompositions. When the Krull dimension of R is at most 1, a derived stratification of R-Mod can be constructed explicitly. Note that an abelian category $\mathcal{A}$ is said to be *derived indecomposable* if it does not have any nontrivial derived decompositions, that is, only $(\mathcal{A}, 0)$ and $(0, \mathcal{A})$ are the derived decompositions of $\mathcal{A}$.

Corollary 4.11. *Suppose that R is a commutative noetherian ring of Krull dimension at most* 1. *Let* $\mathrm{Max}(R)$ *be the set of maximal ideals of R and let* $\mathrm{Min}(R)$ *be the set of prime ideals of R which are not maximal. Then*

(1) $(\mathrm{Supp}^{-1}(\mathrm{Max}(R)), \mathrm{Supp}^{-1}(\mathrm{Min}(R)))$ *is a derived decomposition of* R-Mod.

(2) *There are equivalences of abelian categories*:

$$\mathrm{Supp}^{-1}(\mathrm{Max}(R)) \xrightarrow{\simeq} \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \mathrm{Supp}^{-1}(\{\mathfrak{m}\}),$$

$$\mathrm{Supp}^{-1}(\mathrm{Min}(R)) \xrightarrow{\simeq} \prod_{\mathfrak{p}\in\mathrm{Min}(R)} R_{\mathfrak{p}}\text{-Mod},$$

where $\prod$ *denotes the direct product of abelian categories.*

(3) *Both* $\mathrm{Supp}^{-1}(\{\mathfrak{m}\})$ *and* $R_{\mathfrak{p}}$-Mod *are derived indecomposable for any* $\mathfrak{m} \in \mathrm{Max}(R)$ *and* $\mathfrak{p} \in \mathrm{Min}(R)$.

Proof. (1) Clearly, $\mathrm{Max}(R)$ is a specialization closed subset of $\mathrm{Spec}(R)$. Since the Krull dimension of R is at most 1, the statement (1) follows from Corollary 1.4.

(2) To show the equivalences, we first establish the following

Lemma 4.12. *Let Φ be a subset of* $\mathrm{Spec}(R)$ *with* $\dim(\Phi) \leq 0$. *Then there is an equivalence of abelian categories*: $\prod_{\mathfrak{p}\in\Phi} \mathrm{Supp}^{-1}(\{\mathfrak{p}\}) \xrightarrow{\simeq} \mathrm{Supp}^{-1}(\Phi)$.

Proof. In fact, given an R-module M with a minimal injective resolution $0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$, we always have $\mathrm{Supp}(M) = \bigcup_{i\geq 0} \mathrm{Ass}(I_i) = \bigcup_{i\geq 0} \mathrm{Supp}(I_i)$. Consequently, $M \in \mathrm{Supp}^{-1}(\Phi)$ if and only if $I_i \in \mathrm{Supp}^{-1}(\Phi)$ for all $i \geq 0$. Recall that, for each $\mathfrak{a} \in \mathrm{Spec}(R)$, if $\mathfrak{a} \nsubseteq \mathfrak{b} \in \mathrm{Spec}(R)$, then $\mathrm{Hom}_R(E(R/\mathfrak{a}), E(R/\mathfrak{b}) = 0$. Moreover, $R_{\mathfrak{a}}$-Mod is regarded as an abelian full subcategory of R-Mod and $E(R/\mathfrak{a}) \in R_{\mathfrak{a}}$-Mod. So, the condition $\dim(\Phi) \leq 0$ (that is, $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mathfrak{p} = \mathfrak{q}$ for $\mathfrak{p}, \mathfrak{q} \in \Phi$) implies

(1) $M \in \mathrm{Supp}^{-1}(\Phi)$ if and only if $M \simeq \bigoplus_{\mathfrak{p}\in\Phi} M_{\mathfrak{p}}$ with $M_{\mathfrak{p}} \in \mathrm{Supp}^{-1}(\{\mathfrak{p}\})$, where $M_{\mathfrak{p}}$ stands for the localization of M at $\mathfrak{p}$; and

(2) if $\mathfrak{p}, \mathfrak{q} \in \Phi$ and $\mathfrak{p} \neq \mathfrak{q}$, then $\mathrm{Hom}_R(X, Y) = 0$ for all $X \in \mathrm{Supp}^{-1}(\{\mathfrak{p}\})$ and $Y \in \mathrm{Supp}^{-1}(\{\mathfrak{q}\})$.

By (1) and (2), the functor $\bigoplus : \prod_{\mathfrak{p}\in\Phi} \mathrm{Supp}^{-1}(\{\mathfrak{p}\}) \to \mathrm{Supp}^{-1}(\Phi)$, given by taking direct sums in R-Mod, is an equivalence of abelian categories. □

Now, since the Krull dimension of R is at most 1, we have $\dim(\mathrm{Max}(R)) \leq 0$ and $\dim(\mathrm{Min}(R)) \leq 0$. Moreover, if $\mathfrak{p}$ is a minimal prime ideal of R, then it follows from $\mathrm{Supp}(M) = \bigcup_{i \geq 0} \mathrm{Ass}(I_i)$ that $\mathrm{Supp}^{-1}(\{\mathfrak{p}\}) = R_{\mathfrak{p}}$-Mod. Note that $\mathrm{Min}(R)$ consists of all minimal prime ideals of R which are not maximal. Now, the existence of equivalences in Corollary 4.11 follows from Lemma 4.12.

(3) If $(\mathcal{X}, \mathcal{Y})$ is an Ext-orthogonal pair of an abelian category $\mathcal{A}$ with arbitrary direct sums, then $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}$ is closed under both extensions and arbitrary direct sums (see the dual of [Mac Lane 1998, Theorem 1, p. 116]). It follows from [Krause 2008, Theorem 3.1] that, for any $\mathfrak{a} \in \mathrm{Spec}(R)$, the abelian category $\mathrm{Supp}^{-1}(\{\mathfrak{a}\})$ does not contain nontrivial abelian full subcategory which is closed under extensions and arbitrary direct sums. This implies that the abelian $\mathrm{Supp}^{-1}(\{\mathfrak{a}\})$ is derived indecomposable. Thus (3) holds. □

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